Example 1

A Health Agency has to decide about opening new ambulatories to serve a set of $m$ users ($i = 1, \ldots, m$). A preliminary investigation suggests a set of $n$ potential locations. Each user can be served by an ambulatory on location $j$ ($j = 1, \ldots, n$) if the location is not “too far” from the user’s home. Thus, we are given a matrix $A$ whose entry $a_{ij}$ has value 1 if user $i$ can be served by an ambulatory open in location $j$ and 0 otherwise. The cost of opening an ambulatory on location $j$ is equal to $c_j$. Moreover, the ambulatory in location $j$ can serve at most $M_j$ users. Finally, for planning reasons, the Health Agency wants to open at least $P$ ambulatories, serve all users and minimize the overall cost.

Integer Linear Programming Model:

$$x_{ij} := \begin{cases} 1 & \text{if user } i \text{ is served by ambulatory } j \\ 0 & \text{otherwise} \end{cases} \quad i = 1, \ldots, m; j = 1, \ldots, n;$$

$$y_j := \begin{cases} 1 & \text{if ambulatory } j \text{ is open} \\ 0 & \text{otherwise} \end{cases} \quad j = 1, \ldots, n;$$

$$\min \sum_{j=1}^{n} c_j y_j$$

(1a)

$$\sum_{j=1}^{n} a_{ij} x_{ij} = 1 \quad \forall i = 1, \ldots, m$$

(1b)

$$\sum_{i=1}^{m} x_{ij} \leq M_j y_j \quad \forall j = 1, \ldots, n$$

(1c)

$$\sum_{j=1}^{n} y_j \geq P$$

(1d)

$$y_j \in \{0, 1\} \quad \forall j = 1, \ldots, n$$

(1e)

$$x_{ij} \in \{0, 1\} \quad \forall i = 1, \ldots, m; \forall j = 1, \ldots, n$$

(1f)
Example 2

The Ministry of Home Security must decide concerning the opening of new Fire Stations so as to serve $q$ neighborhoods ($i = 1, \ldots, q$) in Bologna. A preliminary investigation led to the consider a set of $s$ potential areas $j, j = 1, \ldots, s$ where a Fire Station can be built. Each neighborhood $i$ can be served by a Fire Station on area $j$ if that is not “too far”. Then, we are given a binary matrix $A$ whose entry $a_{ij}$ is 1 if neighborhood $i$ can be served by a Fire Station on area $j$ and 0 otherwise. The opening cost for the Fire Station in area $j$ is $r_j$. For security reasons, each neighborhood must be potentially served by at least 2 (open) Fire Stations, and at least $B$ Fire Stations must be open. The constraints must be satisfied by minimizing the overall opening cost.

Integer Linear Programming Model:

\[
y_j := \begin{cases} 
  1 & \text{if Fire Station } j \text{ is open} \\
  0 & \text{otherwise} 
\end{cases} \quad j = 1, \ldots, s; 
\]

\[
\min \sum_{j=1}^{s} r_j y_j \quad (2a)
\]
\[
\sum_{j=1}^{s} a_{ij} y_j \geq 2 \quad \forall i = 1, \ldots, q \quad (2b)
\]
\[
\sum_{j=1}^{s} y_j \geq B \quad (2c)
\]
\[
y_j \in \{0, 1\} \quad \forall j = 1, \ldots, s \quad (2d)
\]

(2e)
Example 3

A pet shop has \( n \) water pools to be used to show a set of \( m \) tropical fishes. Each pool \( j \) has a cost of \( c_j \) and a volume \( V_j \). The shop owner wants to minimize the overall cost of the pools used by considering that, for space reasons, the shop can use at most \( K \) pools simultaneously. It is given a graph \( G = (V, E) \) whose generic edge \( e_{ij} \) defines the incompatibility between fishes \( i \) and \( j \), respectively, of being put in the same pool (because of the natural nutrition chain). Moreover, we are given a binary matrix \( B \) whose entry \( b_{ij} \) is equal to 1 if fish \( i \) can be assigned to pool \( j \) and 0 otherwise. Finally, each fish \( i \) needs a living space of \( v_i \), thus the sum of the living space of the fishes assigned to the same pool \( j \) cannot be higher than the volume of the volume \( V_j \).

Integer Linear Programming Model:

\[
y_j := \begin{cases} 
1 & \text{if pool } j \text{ is used} \\
0 & \text{otherwise} 
\end{cases} \quad j = 1, \ldots, n; \\
x_{ij} := \begin{cases} 
1 & \text{if fish } i \text{ is assigned to pool } j \\
0 & \text{otherwise} 
\end{cases} \quad i = 1, \ldots, m; \quad j = 1, \ldots, n; \\
\min \sum_{j=1}^{n} c_j y_j \\
\sum_{j=1}^{n} y_j \leq K \tag{3a} \\
\sum_{j=1}^{n} b_{ij} x_{ij} = 1 \quad \forall i = 1, \ldots, m \tag{3b} \\
\sum_{i=1}^{m} v_i x_{ij} \leq V_j y_j \quad \forall j = 1, \ldots, n \tag{3c} \\
x_{ij} + x_{hj} \leq y_j \quad \forall (i, h) \in A, j = 1, \ldots, n \tag{3d} \\
x_{ij}, y_j \in \{0, 1\} \quad \forall i = 1, \ldots, m, j = 1, \ldots, n \tag{3e} \\
\end{eqnarray}

Discuss alternatives and/or possible improvements to the above model.