Credit Portfolio Risk

- Introduction to credit portfolio risk.
- CreditMetrics mechanics.
- CreditMetrics through $R$.
- Asset value approach.
- Hints on credit derivatives.
Comparison of Alternative Approaches

<table>
<thead>
<tr>
<th>Approach</th>
<th>Idea</th>
<th>Pros</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>CreditMetrics</td>
<td>Simulation</td>
<td>Traded companies</td>
<td>Estimate</td>
</tr>
<tr>
<td>CreditRisk⁺</td>
<td>Pooling</td>
<td>Default model</td>
<td>Estimate</td>
</tr>
<tr>
<td>CreditPortfolio View</td>
<td>Econometric</td>
<td>Macro-economic</td>
<td>Data</td>
</tr>
</tbody>
</table>

Table: Basic ideas underlying the most popular approaches to credit portfolio.
Merton Model and CreditMetrics

Figure: Merton model compared to CreditMetrics rating threshold approach\(^1\).

\(^1\)Source: CreditMetrics Technical Document.
CreditMetrics Mechanics

It is possible to summarize CreditMetrics approach as follows:

▶ Rating thresholds.
▶ Correlation estimation.
▶ Monte Carlo joint simulation of returns.
▶ For each Monte Carlo run, estimation of portfolio losses.
▶ Loss distribution.
CreditRisk$^+$ Mechanics

It is possible to summarize the default model CreditRisk$^+$ as follows:

- No hypothesis on default source: no structural model as in CreditMetrics.
- Obligors are aggregated into clusters.
- The number of defaults within a cluster is assumed have a Poisson distribution.
- Analytical probability estimation through convolutions.
CreditPortfolio View Mechanics

It is possible to summarize CreditPortfolio View as follows:

- The default probability of obligors is assumed to be
  \[ PD_i = \frac{1}{1 + e^{-Y_i}}. \]  

- \( Y_i \) is an economic health index
  \[ Y_i = \beta_{i,0} + \beta_{i,1}X_1 + \ldots + \beta_{i,p}X_p + \epsilon_i, \]  
  where \( X \) is a vector of macroeconomic variables.

- Simulating the vector \( X \) and \( \epsilon_i \) (assumed to be independent Normally distributed) we obtain the distribution of PD.

- From the distribution of PD it is easy to derive the distribution of losses.
CreditMetrics vs RiskMetrics

The distribution of credit returns is different from that of market returns.

In market risk data to compute correlations are widely available. It is not the same for credit risk.

\(^2\)Source: CreditMetrics Technical Document.
### Bond Value

#### Table: Distribution of value of a BBB par bond in one year

<table>
<thead>
<tr>
<th>Year-end rating</th>
<th>Value ($)</th>
<th>Probability (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>109.37</td>
<td>0.02</td>
</tr>
<tr>
<td>AA</td>
<td>109.19</td>
<td>0.33</td>
</tr>
<tr>
<td>A</td>
<td>108.66</td>
<td>5.95</td>
</tr>
<tr>
<td>BBB</td>
<td>107.55</td>
<td>86.93</td>
</tr>
<tr>
<td>BB</td>
<td>102.02</td>
<td>5.30</td>
</tr>
<tr>
<td>B</td>
<td>98.10</td>
<td>1.17</td>
</tr>
<tr>
<td>CCC</td>
<td>83.64</td>
<td>0.12</td>
</tr>
<tr>
<td>Default</td>
<td>51.13</td>
<td>0.18</td>
</tr>
</tbody>
</table>

#### Figure: Distribution of values for a BBB rated bond

- **Bond rated BBB, fixed rate and maturity 5 years.**
- **The actual value is computed considering the credit risk spread for BBB rating class.**

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3 **Source:** CreditMetrics Technical Document.
Loss Distribution

The loss distribution is obtained comparing the initial value and values for alternative rating classes.

4 Source: CreditMetrics Technical Document.
Two Bonds Portfolio

All possible 64 year-end values for a two-bond portfolio ($S$)

<table>
<thead>
<tr>
<th>Obligor #1 (BBB)</th>
<th>Obligor #2 (single-A)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AAA</td>
</tr>
<tr>
<td>AAA</td>
<td>109.37</td>
</tr>
<tr>
<td>AA</td>
<td>109.19</td>
</tr>
<tr>
<td>A</td>
<td>108.66</td>
</tr>
<tr>
<td>BBB</td>
<td>107.55</td>
</tr>
<tr>
<td>BB</td>
<td>102.02</td>
</tr>
<tr>
<td>B</td>
<td>98.10</td>
</tr>
<tr>
<td>CCC</td>
<td>83.64</td>
</tr>
<tr>
<td>Default</td>
<td>51.13</td>
</tr>
</tbody>
</table>

Joint migration probabilities with zero correlation (%)

<table>
<thead>
<tr>
<th>Obligor #1 (BBB)</th>
<th>Obligor #2 (single-A)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>AAA</td>
</tr>
<tr>
<td>AAA</td>
<td>0.02</td>
</tr>
<tr>
<td>AA</td>
<td>0.33</td>
</tr>
<tr>
<td>A</td>
<td>0.01</td>
</tr>
<tr>
<td>BBB</td>
<td>81.93</td>
</tr>
<tr>
<td>BB</td>
<td>5.30</td>
</tr>
<tr>
<td>B</td>
<td>1.17</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12</td>
</tr>
<tr>
<td>Default</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Figure: Distribution of losses for a BBB rated bond\(^5\).

- It is evident the need to compute correlations.

\(^5\)Source: CreditMetrics Technical Document.
Rating Thresholds

<table>
<thead>
<tr>
<th>Rating</th>
<th>Probability from the transition matrix ($)</th>
<th>Probability according to the asset value model</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.03</td>
<td>$1 - \Phi(Z_{AA}/\sigma)$</td>
</tr>
<tr>
<td>AA</td>
<td>0.14</td>
<td>$\Phi(Z_{A}/\sigma) - \Phi(Z_{AA}/\sigma)$</td>
</tr>
<tr>
<td>A</td>
<td>0.67</td>
<td>$\Phi(Z_{A}/\sigma) - \Phi(Z_{BBB}/\sigma)$</td>
</tr>
<tr>
<td>BBB</td>
<td>7.73</td>
<td>$\Phi(Z_{BBB}/\sigma) - \Phi(Z_{BB}/\sigma)$</td>
</tr>
<tr>
<td>BB</td>
<td>80.53</td>
<td>$\Phi(Z_{BB}/\sigma) - \Phi(Z_{B}/\sigma)$</td>
</tr>
<tr>
<td>B</td>
<td>8.84</td>
<td>$\Phi(Z_{B}/\sigma) - \Phi(Z_{CCC}/\sigma)$</td>
</tr>
<tr>
<td>CCC</td>
<td>1.00</td>
<td>$\Phi(Z_{CCC}/\sigma) - \Phi(Z_{DEF}/\sigma)$</td>
</tr>
<tr>
<td>Default</td>
<td>1.06</td>
<td>$\Phi(Z_{DEF}/\sigma)$</td>
</tr>
</tbody>
</table>

\[ Z_{DEF} = \Phi^{-1}(1.06\%) \cdot \sigma = -2.30\sigma \]

Figure: Transition probabilities and asset value model\(^6\).

- For large portfolios it is useful to consider Monte Carlo simulations.

\(^6\) Source: CreditMetrics Technical Document.
Correlations

There are alternative ways to estimate correlations:

- From historical asset values.
- From credit spreads (bonds, CDS, ...).
- From macroeconomic factors considering sensitivity weights.
- From market share values considering sensitivity weights (i.e. CreditMetrics).
CreditMetrics Example 1/2

### Transition Probabilities (%)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Firm 1</th>
<th>Firm 2</th>
<th>Firm 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.02</td>
<td>0.09</td>
<td>0.22</td>
</tr>
<tr>
<td>AA</td>
<td>0.33</td>
<td>2.27</td>
<td>0.00</td>
</tr>
<tr>
<td>A</td>
<td>5.95</td>
<td>91.05</td>
<td>0.22</td>
</tr>
<tr>
<td>BBB</td>
<td>86.93</td>
<td>5.52</td>
<td>1.30</td>
</tr>
<tr>
<td>BB</td>
<td>5.30</td>
<td>0.74</td>
<td>2.38</td>
</tr>
<tr>
<td>B</td>
<td>1.17</td>
<td>0.26</td>
<td>11.24</td>
</tr>
<tr>
<td>CCC</td>
<td>0.12</td>
<td>0.01</td>
<td>64.86</td>
</tr>
<tr>
<td>Default</td>
<td>0.18</td>
<td>0.06</td>
<td>19.79</td>
</tr>
</tbody>
</table>

### Asset Return Thresholds

<table>
<thead>
<tr>
<th>Threshold</th>
<th>Firm 1</th>
<th>Firm 2</th>
<th>Firm 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_{AA}$</td>
<td>3.54</td>
<td>3.12</td>
<td>2.86</td>
</tr>
<tr>
<td>$Z_{A}$</td>
<td>2.78</td>
<td>1.98</td>
<td>2.86</td>
</tr>
<tr>
<td>$Z_{BBB}$</td>
<td>1.53</td>
<td>-1.51</td>
<td>2.63</td>
</tr>
<tr>
<td>$Z_{BB}$</td>
<td>-1.49</td>
<td>-2.30</td>
<td>2.11</td>
</tr>
<tr>
<td>$Z_{B}$</td>
<td>-2.18</td>
<td>-2.72</td>
<td>1.74</td>
</tr>
<tr>
<td>$Z_{CCC}$</td>
<td>-2.75</td>
<td>-3.19</td>
<td>1.02</td>
</tr>
<tr>
<td>$Z_{Def}$</td>
<td>-2.91</td>
<td>-3.24</td>
<td>-0.85</td>
</tr>
</tbody>
</table>

### Correlation Matrix for Example Portfolio

<table>
<thead>
<tr>
<th></th>
<th>Firm 1</th>
<th>Firm 2</th>
<th>Firm 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>1.0</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>Firm 2</td>
<td>0.3</td>
<td>1.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Firm 3</td>
<td>0.1</td>
<td>0.2</td>
<td>1.0</td>
</tr>
</tbody>
</table>

---

**Figure:** Ingredients for CreditMetrics loss distribution\(^7\).

\(^7\)Source: CreditMetrics Technical Document.
### Mapping return scenarios to rating scenarios

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Asset Return</th>
<th>New Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm 1</td>
<td>Firm 2</td>
</tr>
<tr>
<td>1</td>
<td>-0.7769</td>
<td>-0.8750</td>
</tr>
<tr>
<td>2</td>
<td>-2.1060</td>
<td>-2.0646</td>
</tr>
<tr>
<td>3</td>
<td>-0.9276</td>
<td>0.0606</td>
</tr>
<tr>
<td>4</td>
<td>0.6454</td>
<td>-0.1532</td>
</tr>
<tr>
<td>5</td>
<td>0.4690</td>
<td>-0.5639</td>
</tr>
<tr>
<td>6</td>
<td>-0.1252</td>
<td>-0.5570</td>
</tr>
<tr>
<td>7</td>
<td>0.6994</td>
<td>1.5191</td>
</tr>
<tr>
<td>8</td>
<td>1.1778</td>
<td>-0.6342</td>
</tr>
<tr>
<td>9</td>
<td>1.3480</td>
<td>2.1202</td>
</tr>
<tr>
<td>10</td>
<td>0.0249</td>
<td>-0.4642</td>
</tr>
</tbody>
</table>

### Valuation of portfolio scenarios (Smm)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Rating</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm 1</td>
<td>Firm 2</td>
</tr>
<tr>
<td>1</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>BB</td>
<td>BBB</td>
</tr>
<tr>
<td>3</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>4</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>6</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>7</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>8</td>
<td>BBB</td>
<td>A</td>
</tr>
<tr>
<td>9</td>
<td>A</td>
<td>AA</td>
</tr>
<tr>
<td>10</td>
<td>BBB</td>
<td>A</td>
</tr>
</tbody>
</table>

**Figure:** Portfolio valuation.

---

8 **Source:** CreditMetrics Technical Document.
**cm.cs Credit Spread**

*cm.cs* computes the credit spreads for each rating of a one year empirical migration matrix. The failure limit is the quantile of the failure probability.

- **Usage** *cm.cs(M, lgd)*
  
  M one year empirical migration matrix, where the last row gives the default class.
  lgd loss given default.

- *cm.cs* return value is the credit spread for time $t = 1$ of each rating in the migration matrix.
### Transition Matrix and cm.cs Credit Spread Output

```r
lgd <- 0.45
M <- matrix(c(
90.81, 8.33, 0.68, 0.06, 0.08, 0.02, 0.01, 0.01,
0.70, 90.65, 7.79, 0.64, 0.06, 0.13, 0.02, 0.01,
0.09, 2.27, 91.05, 5.52, 0.74, 0.26, 0.01, 0.06,
0.02, 0.33, 5.95, 85.93, 5.30, 1.17, 1.12, 0.18,
0.03, 0.14, 0.67, 7.73, 80.53, 8.84, 1.00, 1.06,
0.01, 0.11, 0.24, 0.43, 6.48, 83.46, 4.07, 5.20,
0.21, 0, 0.22, 1.30, 2.38, 11.24, 64.86, 19.79,
0, 0, 0, 0, 0, 0, 0, 100
)/100, 8, 8, dimnames=list(csv, csv), byrow=TRUE)
```

<table>
<thead>
<tr>
<th></th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>LGD</td>
<td>4.500101e-05</td>
<td>4.500101e-05</td>
<td>2.700365e-04</td>
<td>8.103282e-04</td>
</tr>
<tr>
<td>GB</td>
<td>4.781413e-03</td>
<td>2.367813e-02</td>
<td>9.327276e-02</td>
<td></td>
</tr>
</tbody>
</table>
**cm.ref Reference Value**

*cm.ref* computes the value of a credit in one year for each rating, this is the return value *constVal*. Further the portfolio value at time $t = 1$ is computed, this is *constPV*.

- **Usage** *cm.ref*(M, lgd, ead, r, rating)
  - M one year empirical migration matrix.
  - lgd loss given default.
  - ead exposure at default.
  - r riskless interest rate.
  - rating rating of companies.

- **Details**

  $$V_t = EAD_t e^{-(r_t + CS_t)t}$$  \hspace{1cm} (3)

- *cm.ref* returns a list containing following components:
  - *constVal* credit value in one year ($t = 1$).
  - *constPV* portfolio of all credit values in one year ($t = 1$).
**cm.ref Output**

\[ r \leftarrow 0.03 \]
\[ \text{ead} \leftarrow c(40, 100, 200) \]
\[ \text{rating} \leftarrow c("BBB", "AA", "B") \]
\[ \text{ref.val} \leftarrow \text{cm.ref}(M, \text{lgd}, \text{ead}, r, \text{rating}) \]

#
\[ \text{ref.val}$\text{constVal} \]

- BBB: 38.78638
- AA: 97.04019
- B: 189.54742

#
\[ \text{ref.val}$\text{constPV} \]

325.374
**cm.state Rating State Space**

*cm.state* computes a state space, this is at time $t = 1$ the credit positions of all companies for all migrations is calculated. This state space is needed for the later valuation for the credit positions of each scenario.

- **Usage**
  
  $cm.state(M, lgd, ead, N, r)$
  
  - $M$: one year empirical migration matrix.
  - $lgd$: loss given default.
  - $ead$: exposure at default.
  - $N$: number of companies.
  - $r$: riskless interest rate.

- **cm.state** return value is the matrix $V$ for time $t = 1$ of each rating in the migration matrix including the credit values for all companies. The last column in the matrix $V$ is the value for the default event of each company.
### cm.state Output

```r
state.space <- cm.state(M, lgd, ead, N, r)
#
# AAA   AA   A    BBB
[1,] 38.81607 38.81607 38.80734 38.78638
[2,] 97.04019 97.04019 97.01835 96.96595
[3,] 194.08037 194.08037 194.03670 193.93189
#
# BB   B    CCC   D
[1,] 38.63266 37.90948 35.36090 22
[2,] 96.58165 94.77371 88.40225 55
[3,] 193.16330 189.54742 176.80450 110
```
**cm.rnorm.cor** Correlated Normal Random Numbers

*cm.rnorm.cor* computes correlated standard normal distributed random numbers. This function uses a correlation matrix *rho* and later the Cholesky decomposition in order to get correlated random numbers.

**Usage** *cm.rnorm.cor*(N, n, rho)

- **N** number of companies.
- **n** number of simulated random numbers.
- **rho** correlation matrix.

The function returns *N* simulations with *n* simulated random numbers each, which include the correlation matrix *rho*. 
N <- 3
n <- 4
firmnames <- c("firm BBB", "firm AA", "firm B")
rho <- matrix(c( 1.0, 0.4, 0.6,
                 0.4, 1.0, 0.5,
                 0.6, 0.5, 1.0), 3, 3,
                 dimnames = list(firmnames, firmnames),byrow = TRUE)
rand.cor<-cm.rnorm.cor(N, n, rho)

firm BBB -1.131922  0.0634755  1.131922  -0.0634755
firm AA   0.836114  -0.1660624 -0.836114   0.1660624
firm B   -1.355125  0.5741706  1.355125  -0.5741706
**cm.quantile** Migration Quantiles

*cm.quantile* computes the empirical migration quantiles for each rating of a one year empirical migration matrix. The failure limit is the quantile of the failure probability.

**Usage**

\[ cm.quantile(M) \]

- **M** one year empirical migration matrix.

**Details**

\[ S = N^{-1}(PD) \]  

(4)

- The function returns the quantile of each rating in the migration matrix.
## cm.quantile Output

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>CCC</th>
<th>B</th>
<th>BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>-Inf</td>
<td>-3.7190165</td>
<td>-3.540084</td>
<td>-3.352795</td>
</tr>
<tr>
<td>AA</td>
<td>-Inf</td>
<td>-3.7190165</td>
<td>-3.431614</td>
<td>-2.947843</td>
</tr>
<tr>
<td>A</td>
<td>-Inf</td>
<td>-3.2388801</td>
<td>-3.194651</td>
<td>-2.716381</td>
</tr>
<tr>
<td>BBB</td>
<td>-Inf</td>
<td>-2.9112377</td>
<td>-2.226212</td>
<td>-1.965123</td>
</tr>
<tr>
<td>BB</td>
<td>-Inf</td>
<td>-2.3044036</td>
<td>-2.041512</td>
<td>-1.231864</td>
</tr>
<tr>
<td>B</td>
<td>-Inf</td>
<td>-1.6257634</td>
<td>-1.324310</td>
<td>1.455973</td>
</tr>
<tr>
<td>CCC</td>
<td>-Inf</td>
<td>-0.8491461</td>
<td>1.021537</td>
<td>1.738061</td>
</tr>
</tbody>
</table>

### BBB

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>AA</th>
<th>AAA</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>-2.911238</td>
<td>-2.382404</td>
<td>-1.329145</td>
</tr>
<tr>
<td>AA</td>
<td>-2.382404</td>
<td>-1.362627</td>
<td>2.457263</td>
</tr>
<tr>
<td>A</td>
<td>-1.507042</td>
<td>1.984501</td>
<td>3.121389</td>
</tr>
<tr>
<td>BBB</td>
<td>1.530068</td>
<td>2.696844</td>
<td>3.540084</td>
</tr>
<tr>
<td>BB</td>
<td>2.391056</td>
<td>2.929050</td>
<td>3.431614</td>
</tr>
<tr>
<td>B</td>
<td>2.687449</td>
<td>3.035672</td>
<td>3.719016</td>
</tr>
<tr>
<td>CCC</td>
<td>2.627559</td>
<td>2.862736</td>
<td>2.862736</td>
</tr>
</tbody>
</table>
**cm.val Valuation of Each Scenario**

`cm.val` performs a valuation for the credit positions of each scenario. This is an allocation in rating classes identification of the credit position values.

- **Usage**
  
  ```r
  cm.state(M, lgd, ead, N, r)
  ```

  - `M` one year empirical migration matrix.
  - `lgd` loss given default.
  - `ead` exposure at default.
  - `N` number of companies.
  - `n` number of simulated random numbers.
  - `r` riskless interest rate.
  - `rho` correlation matrix.
  - `rating` rating of companies.

- **cm.val** returns simulated values of the firms for each rating of each scenario.
The distribution depends on the random numbers generated through \textit{cm.rnorm.cor} and the link between the threshold computed using \textit{cm.quantile} and the states obtained from \textit{cm.state}.
**cm.gain** Profits and Losses

*cm.gain* computes profits or losses, this is done by building the difference from the reference value and the simulated portfolio values of the credit positions.

- **Usage**  
  \[ cm.gain(M, lgd, ead, N, n, r, rho, rating) \]  
  - \(M\): one year empirical migration matrix.  
  - \(lgd\): loss given default.  
  - \(ead\): exposure at default.  
  - \(N\): number of companies.  
  - \(n\): number of simulated random numbers.  
  - \(r\): riskless interest rate.  
  - \(rho\): correlation matrix.  
  - \(rating\): rating of companies.

- *cm.gain* returns simulated profits or losses.
**cm.gain Output**

```r
gain<-cm.gain(M, lgd, ead, N, n, r, rho, rating)
#
[1]  -134.54743  -1E-05   -1E-05   -1E-05
```

- Gain is the difference between `cm.val` and `cm.ref` (referred to the entire portfolio).
CreditMetrics Through R

**cm.gain Plot**

- `n=20000`

```r
gain<-cm.gain(M, lgd, ead, N, n, r, rho, rating)
hist.gain<-hist(gain, col="steelblue4",
main="Profit/Loss Distribution",
xlab="profit/loss", ylab="frequency")
```

![Profit / Loss Distribution](image)
A Simplified Approach

We focus on a simplified framework in which we just consider losses from default (but not from changes in market value).

1. Specify probabilities of individual credit events (PD) as other events (changes in credit quality) are ignored in the modeling.
2. Specify value effects of individual credit events: loss given default (LGD). It is the percentage of exposure at default (EAD) that is lost in case of default.
3. Specify correlations of individual credit events and value effects
4. Based on steps 1 to 3, obtain the portfolio value distribution.
Default Threshold

There are different ways to obtain PDs as well as LGDs. In what follows we focus on the third step choosing to employ the asset value approach to define the default event.

▶ The asset value model represents default correlations by linking defaults to a continuous variable, the asset value $A$. Borrower $i$ defaults if its asset value falls below some threshold $d_i$ chosen to match the specified $PD_i$ as follows

$$1_{A \leq d_i} = \begin{cases} 1 & \text{for } A \leq d_i \\ 0 & \text{for } A > d_i. \end{cases}$$ (5)

▶ If the asset values are assumed to be standard normally distributed, we would set $d_i = \Phi^{-1}(PD_i)$, where $\Phi$ denotes the cumulative standard normal distribution function.
Factor Model

- Correlation in asset values can be modeled through factor models. We start with a simple one containing just one systematic factor $Z$ as follows

\[
A_i = w_i Z + \sqrt{1 - w_i^2} \epsilon_i
\]

\[
\text{cov}(\epsilon_i, \epsilon_j) = 0 \quad i \neq j
\]

\[
\text{cov}(Z, \epsilon_i) = 0 \quad \forall i
\]

\[
Z \sim N(0, 1)
\]

\[
\epsilon_i \sim N(0, 1), \quad \forall i.
\]

- Systematic ($Z$) and idiosyncratic ($\epsilon$) shocks are independent.
- Idiosyncratic shocks deserve their name because they are independent across firms.
- Shocks are standard normally distributed.
Factor Model

In the asset value approach, the standard way of obtaining the portfolio distribution (step 4) is to run a Monte Carlo simulation. It has the following structure.

1. Randomly draw asset values for each obligor in the portfolio (which we will do here according to equation (6)).
2. For each obligor, check whether it defaulted according to (5). If yes, determine the individual loss.
3. Aggregate the individual losses into a portfolio loss.
4. Repeat steps 1 to 3 sufficiently often to arrive at a distribution of credit portfolio losses.
Simulation Procedure Adjustment

Since we are concerned with large losses, let us first state how such large losses can come about.

- Recall that default occurs if the asset value $A_i$ drops below the default threshold, and that we modeled $A_i$ as
  $$A_i = w_i Z + \sqrt{1 - w_i^2} \epsilon_i.$$  
- There are two situations in which the number of defaults is large (they can, of course, come about at the same time).
  - The factor realization $Z$ is negative (think of the economy moving into a recession).
  - The average $\epsilon_i$ is negative (think of many firms having individual bad luck).
  - The larger the number of obligors in a portfolio, and the more even are the exposures distributed across obligors, the more important will be the first effect relative to the second.
### Shifted Mean

- To tilt our simulation towards scenarios with large losses, we can instead sample the factor from a normal distribution with mean $\mu < 0$, leaving the standard deviation at 1.

- When modeling correlations through the one factor model, we assumed the factor to have a mean of zero, but now we work with a mean different from zero. There is a quick way of correcting this bias, however.

- Before importance sampling, the probability of observing a trial $j$ is just $1/M$, where $M$ is the chosen number of trials. With importance sampling, we get the trial probability by multiplying $1/M$ with the likelihood ratio

$$\frac{\phi(Z_j)}{\phi(Z_j - \mu)}$$

where $\phi$ is the standard normal density, and $\mu$ is the mean of $Z$ assumed in the importance sampling.
Importance Sampling Distribution

When implementing importance sampling, it is useful to note that
\[
\frac{\phi(Z_j)}{\phi(Z_j - \mu)} = \frac{(2\pi)^{-1/2} \exp(-Z_j^2/2)}{(2\pi)^{-1/2} \exp(-(Z_j - \mu)^2/2)} = \exp(-\mu Z_j + \mu^2/2) \quad (8)
\]

The probability of observing the loss of trial \(j\) is therefore
\[
pr_j = \exp(-\mu Z_j + \mu^2/2)/M \quad (9)
\]

Starting from the largest loss of the sorted simulated vector, cumulate the trial probabilities (9). Determine the percentile as the maximum loss that has a cumulated probability larger than \((1 - \alpha)\).
Credit Default Swap Definition and Mechanics

- Definition of Credit Default Swap: CDS. In a CDS contract one party (Protection Buyer: PB) agrees to make periodic payments to the other party (Protection Seller: PS) in exchange of protection against a credit event (default) with respect to an underlying entity (name).

Deutsche Bank  
(Name)

Default Swap Spread: \( s \)  
Notional: \( N \)  
Time period: \( \Delta_i \)  
Cash Flows at each time \( T_i: sN \Delta_i \)

Unicredit  
(Protection Buyer)

Capitalia  
(Protection Seller)

Notional: \( N \)  
Recovery Rate: \( R \)  
Default time: \( \tau \)  
Cash Flow at \( \tau \): \((1-R)N\)

\( T_0 \)  
\( T_1 \)  
\( T_2 \)  
...  
\( T \)  
\( \tau \) (default time)
Basket Default Swap Definition and Mechanics

- A basket default swap is like a credit default swap where the credit event is the default of some combination of the credits in a basket of names.

```
Deutsche Bank,
General Electric,
Intesa Bank
(Names)

Unicredit
(Protection Buyer)

Default Swap Spread: s
Notional: N
Time period: Δ_i
Cash Flows at each time T_j: sN Δ_j

Capitalia
(Protection Seller)

Notional: N
Recovery Rate: R
Time of the First, Second or Third default: τ
Cash Flow at τ: (1-R)N

T_0       T_1       T_2       ...       T       τ (default time)
```

Need to Consider the Correlation Among the Default of All Names
CDS and Basket Default Swap Pricing

- Equilibrium at Inception

\[ CDS^{PB}(t, T|\tau) = CDS^{PS}(t, T|\tau). \] (10)

- Protection Buyer Leg

\[ CDS^{PB}(t, T|\tau) = \sum_{i=1}^{k} sN\Delta_i 1_{\tau>T_i} \cdot D(t, T_i). \] (11)

- Protection Seller Leg

\[ CDS^{PS}(t, T|\tau) = (1 - R)N 1_{\tau\leq T} \cdot D(t, \tau). \] (12)

- Default Swap Spread

\[ s = \frac{(1 - R)1_{\tau\leq T} \cdot D(t, \tau)}{\sum_{i=1}^{k} \Delta_i 1_{\tau>T_i} \cdot D(t, T_i)}. \] (13)
Pricing Zero Coupon Bonds

- Default Free Zero Coupon Bond Pricing

\[ V(t, T) = E \left[ e^{- \int_t^T r(u) du} \cdot 1 \right]. \] (14)

where \( r(\cdot) \) is the stochastic default free interest rate.

- Defaultable Zero Coupon Bond Pricing

\[ \tilde{V}(t, T) = E \left[ e^{- \int_t^T r(u) du} \cdot 1_{\tau > T} \right] \] (15)

and, assuming independence between interest rates and default time dynamics, we have:

\[ \tilde{V}(t, T) = E \left[ e^{- \int_t^T r(u) du} \cdot 1_{\tau > T} \right] = E \left[ e^{- \int_t^T r(u) du} \right] E \left[ 1_{\tau > T} \right] = V(t, T) E \left[ 1_{\tau > T} \right] = V(t, T) S(t, T) \] (16)

where \( E \left[ 1_{\tau > T} \right] = S(t, T) \) is the survival probability of the defaultable firm.
Intensity of Default

Starting from $S(\cdot)$ we obtain intensity of default $\lambda(\cdot)$ as follows

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{S(t) - S(t + \Delta t)}{\Delta t S(t)} = -\frac{1}{S(t)} \lim_{\Delta t \to 0} \frac{S(t + \Delta t) - S(t)}{\Delta t} = -\frac{S'(t)}{S(t)}$$

(17)

where $S'(t)$ is the first derivative of $S(t)$ with respect to $t$.

We can represent intensity of default even from the cumulative point of view. Conventionally, the cumulative function from $t$ to $T$ is denoted as $\Lambda(t, T)$ and

$$\Lambda(t, T) = -\ln [S(t, T)].$$

(18)

If we assume that the survival function $S(\cdot)$ is exponentially distributed we can state that

$$\Lambda(t, T) = -\ln [S(t, T)] = -\ln \left[ e^{-\int_t^T \lambda(u) du} \right] = \int_t^T \lambda(u) du.$$

(19)
**Default Time** \( \tau \)

- We state that default occurs if the survival probability \( S(t, T) \leq U \), where \( U \) is a uniform random variable. We need to estimate \( \lambda(\cdot) \) in order to compute \( S(t, T) \), then we generate uniform random numbers determining whether \( S(t, T) \leq U \) (default) or \( S(t, T) > U \) (survival). Considering that, in our setting, the function \( S(t, T) \) is as follows

\[
S(t, T) = e^{-\int_t^T \lambda(u)du},
\]

default occurs if

\[
\int_t^T \lambda(u)du \geq -\ln U.
\]

- We can alternatively state that default time \( \tau \) is

\[
\tau = \inf \left\{ \text{time} : \int_t^{\text{time}} \lambda(u)du \geq Q \right\}
\]

where \( Q \) is an exponential random variable with parameter 1.
Basket Default Swap Pricing

- **Survival Probability.** Compute, for each name \( i = 1, \ldots, n \), starting from real market datasets, the integral \( \int_t^T \lambda_i(u) du \) considering \( \lambda_i(u) \) as piecewise constant.

- **Simulation of Uniform Variates with Copulas.** Simulate \( n \) dimensional vector \( u = (u_1, \ldots, u_n)' \) of uniform variates from a copula \( C \) with parameter estimated from a real market dataset.

- **Unit Mean Exponential Random Variable.** Compute the unit mean exponential random variable \( Q \) of Equation (22) as \(-\ln(u_i)\) for \( i = 1, \ldots, n \).

- **Default Time.** Compare \(-\ln(u_i)\) and \( \int_t^T \lambda_i(u) du \) in order to define the time of default as

\[
\tau = \inf \left\{ \text{time} : \int_t^{\text{time}} \lambda(u) du \geq Q \right\}.
\]
Conclusions

- We introduced credit portfolio analysis.
- We analyzed in more detail CreditMetrics even through $R$ software.
- Emphasis has been devoted to correlation distinguishing among: asset, default and default rate correlation.
- We introduced intensity models paying attention to credit derivatives pricing.
References


References


