A (too) short introduction to Fourier Transform

Let $f$ a real function of a real variable. The Fourier Transform (FT) of $f$ is the complex valued function:

$$\mathcal{F}f(s) := \int_{-\infty}^{+\infty} e^{-2\pi ist} f(t)dt$$

For now, just take this as a formal definition; we will not discuss when such an integral exists. One value is easy to compute, and worth pointing out, namely for $s = 0$ we have

$$\mathcal{F}f(0) := \int_{-\infty}^{+\infty} f(t)dt$$

The inverse Fourier transform is defined by:

$$\mathcal{F}^{-1}g(t) := \int_{-\infty}^{+\infty} e^{-2\pi ist} g(s)ds$$

The Fourier inversion theorem states that

$$\mathcal{F}(\mathcal{F}^{-1}g) = g, \quad \mathcal{F}^{-1}(\mathcal{F}f) = f$$

Examples

1. The triangle function. Consider the “triangle function”, defined by $\Lambda(x) = \max\{1 - |x|, 0\}$. Notice that the explicit expression of $\Lambda$ is then

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

![Figure 1: Graph of triangle function](image)

For the Fourier transform we compute (using the fact that sine function is odd):

$$\mathcal{F}\Lambda(s) = \int_{-\infty}^{+\infty} e^{-2\pi ist}\Lambda(t)dt = \int_{-1}^{1} (\cos(2\pi st) - i \sin(2\pi st)) (1 - |t|) dt = \int_{-1}^{1} \cos(2\pi st) (1 - |t|) dt$$

It is worth noting that this is a general fact: for each even function $f(t)(= f(-t))$ the following relation holds

$$\mathcal{F}f(s) = \int_{-\infty}^{+\infty} \cos(2\pi st)f(t)dt = 2\int_{0}^{+\infty} \cos(2\pi st)f(t)dt$$

Moreover if $f(t)(= - f(-t))$ is an odd function, we have

$$\mathcal{F}f(s) = -i \int_{-\infty}^{+\infty} \sin(2\pi st)f(t)dt = -2i\int_{0}^{+\infty} \sin(2\pi st)f(t)dt$$

Let us go back to the FT of the triangle function:

$$\mathcal{F}\Lambda(s) = 2\int_{0}^{1} \cos(2\pi st) (1 - t) dt = 2 \left( \frac{\sin(2\pi s)}{2\pi s} - \frac{2\pi s \sin(2\pi s) + \cos(2\pi s) - 1}{4\pi^2 s^2} \right)$$

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Now, simplifying we get
\[ \mathcal{F}(\Lambda(s)) = \frac{1 - \cos(2\pi s)}{2\pi^2 s^2} \]
The very last step comes from the trigonometrical identity \(1 - \cos(2x) = 2\sin^2 x\)
\[ \mathcal{F}(\Lambda(s)) = \left(\frac{\sin(\pi s)}{\pi s}\right)^2 \]

2. Exponential even function. Let \(f(t) = e^{-|t|}\), then
\[ \mathcal{F}(f(s) = 2 \int_{0}^{+\infty} \cos(2\pi st) f(t) dt = 2 \int_{0}^{+\infty} e^{-t} \cos(2\pi st) dt = \left[ \frac{2e^{-t}(2\pi s \sin(2\pi st) - \cos(2\pi st))}{4\pi^2 s^2 + 1} \right]_{t=0}^{t=+\infty} \]
so that
\[ \mathcal{F}(f(s) = \frac{2}{1 + 4\pi^2 s^2} \]

3. Gaussian function. Let \(f(t) = e^{-\pi t^2}\). Then:
\[ \mathcal{F}(f(s) = \int_{-\infty}^{+\infty} e^{-2\pi ist} e^{-\pi t^2} dt \]
Differentiate with respect to \(s\):
\[ \frac{d}{ds} \mathcal{F}(f(s) = \int_{-\infty}^{+\infty} (-2\pi it) e^{-2\pi ist} e^{-\pi t^2} dt = \int_{-\infty}^{+\infty} (ie^{-\pi t^2})' e^{-2\pi ist} dt \]
so we can integrate by parts obtaining:
\[ \frac{d}{ds} \mathcal{F}(f(s) = -\int_{-\infty}^{+\infty} i e^{-\pi t^2} (-2\pi is) e^{-2\pi ist} dt = -2\pi s \int_{-\infty}^{+\infty} e^{-\pi t^2} e^{-2\pi ist} dt = -2\pi s \mathcal{F}(f(s) \]
In such a way we proved that \(\mathcal{F}(f(s)\) satisfies the differential equation
\[ \frac{d}{ds} \mathcal{F}(f(s) = -2\pi s \mathcal{F}(f(s) \]
whose unique solution, incorporating the initial condition, is
\[ \mathcal{F}(f(s) = \mathcal{F}(f(0)) e^{-\pi s^2} \]
But
\[ \mathcal{F}(f(0) = \int_{-\infty}^{+\infty} e^{-\pi t^2} dt = 1 \]
Hence
\[ \mathcal{F}(f(s) = e^{-\pi s^2} \]
We have found the remarkable fact that the Gaussian \(f(t) = e^{-\pi t^2}\) is its own Fourier transform i.e. a fixed point of the FT.

Properties

1. Linearity One of the simplest and most frequently invoked properties of the Fourier transform is that it is linear (operating on functions). This means:
\[ \mathcal{F}(f + g)(s) = \mathcal{F}(f)(s) + \mathcal{F}(g)(s) \]
\[ \mathcal{F}(\alpha f)(s) = \alpha \mathcal{F}(f)(s) \]
where \(\alpha\) is any number (real or complex).

2. The shift theorem A shift of the variable \(t\) (in many applications a delay in time) has a simple effect on the Fourier transform. To compute the Fourier transform of \(f(t + b)\) for any constant \(b\), we have, let us write \(f_b(t) = f(t + b)\):
\[ \mathcal{F}(f_b)(s) = \int_{-\infty}^{+\infty} f(t + b) e^{-2\pi ist} dt = \int_{-\infty}^{+\infty} f(u) e^{-2\pi is(u-b)} du = e^{2\pi isb} \int_{-\infty}^{+\infty} f(u) e^{-2\pi i su} du = e^{2\pi isb} \mathcal{F}(f)(s) \]

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3. The stretch theorem How does the Fourier transform change if we stretch or shrink the variable in the domain? More precisely, we want to know if we scale $t$ to at what happens to the Fourier transform of $a f(t) = f(at)$. First suppose $a > 0$. Then

$$\mathcal{F}_a f(s) = \frac{1}{a} \mathcal{F} \left( \frac{s}{a} \right)$$

If $a < 0$ the limits of integration are reversed when we make the substitution $u = at$, and so the resulting transform is

$$\mathcal{F}_a f(s) = \frac{1}{|a|} \mathcal{F} \left( \frac{s}{a} \right)$$

Since $-a$ is positive when $a$ is negative, we can combine the two cases and present the Stretch Theorem as:

$$\mathcal{F}_a f(s) = \frac{1}{|a|} \mathcal{F} \left( \frac{s}{a} \right)$$

Exercise Prove that the Fourier Transform of

$$f(t) = \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/(2\sigma^2)}$$

is

$$\mathcal{F}f(s) = e^{-2\pi^2 \sigma^2 s^2}$$

4. Combining shifts and stretches We can combine the shift theorem and the stretch theorem to find the Fourier transform of $f(at + b) = a f_b(t)$

$$\mathcal{F}_a f_b(s) = \frac{1}{|a|} \mathcal{F} \left( \frac{s}{a} \right)$$

Convolution

The convolution of two functions $g(t)$ and $f(t)$ is the function

$$(g * f)(t) = \int_{-\infty}^{+\infty} g(t-x)f(x)dx$$

Observe that $(g * f)(t) = (f * g)(t)$. Moreover the Convolution Theorem holds:

$$\mathcal{F}(g * f)(s) = \mathcal{F}g(s) \mathcal{F}f(s)$$

Another interesting property is:

$$\mathcal{F}(gf)(s) = (\mathcal{F}g * \mathcal{F}f)(s)$$

Ordinary differential equations

The derivative formula To put the Fourier transform to work, we need a formula for the Fourier transform of the derivative

$$\mathcal{F}f'(s) = 2\pi is \mathcal{F}f(s)$$

We see that differentiation has been transformed into multiplication, another remarkable feature of the Fourier transform and another reason for its usefulness. Formulae for higher derivatives also hold, and the result is:

$$\mathcal{F}f^{(n)}(s) = (2\pi is)^n \mathcal{F}f(s)$$

Let us solve, for example, the following ordinary differential equation

$$u'' - u = -f$$

$f(t)$ is a given function and you want to find $u(t)$. Take the Fourier transform of both sides:

$$(2\pi is)^2 \mathcal{F}u - \mathcal{F}u = -\mathcal{F}f$$

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and the solve with respect to \( \mathcal{F}u \)

\[
\mathcal{F}u = \frac{1}{1 + 4\pi^2 s^2} \mathcal{F}f
\]

we recognize \( 1/(1 + 4\pi^2 s^2) \) as the Fourier transform of \( 1/2 e^{-|t|} \), that is,

\[
\mathcal{F}u = \mathcal{F} \left( \frac{1}{2} e^{-|t|} \right) \mathcal{F}f
\]

The right hand side is the product of two Fourier transforms. Therefore, according to the convolution theorem,

\[
u(t) = \frac{1}{2} e^{-|t|} \ast f(t)
\]

Written out in full this is

\[
u(t) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|t-\tau|} f(\tau) d\tau
\]

The heat equation

Besides this equation arises in Mathematical Physics it is of interest in Mathematical Finance also. We deal with the partial differential equation

\[
\begin{aligned}
&\frac{\partial u}{\partial t}(x, t) = u_{xx}(x, t) \quad \text{for } t > 0, \ x \in \mathbb{R} \\
&u(x, 0) = f(x) \quad \text{for } t = 0, \ x \in \mathbb{R}
\end{aligned}
\]

Both \( f(x) \) and \( u(x, t) \) are defined for \(-\infty < x < +\infty\). Knowing the Fourier transform of the Gaussian is essential for the treatment we are about to give. The idea is to take the Fourier transform of both sides of the heat equation, “with respect to \( x \)” thinking \( t \) as a “parameter”. The Fourier transform of the right hand side of the equation, \( u_{xx}(x, t) \), is

\[
\mathcal{F}u_{xx}(s, t) = (2\pi is)^2 \mathcal{F}u(s, t) = -4\pi^2 s^2 \mathcal{F}u(s, t)
\]

For the left hand side, \( u_t(x, t) \), we do something different. We have

\[
\mathcal{F}u_t(s, t) = \int_{-\infty}^{+\infty} u_t(x, t)e^{-2\pi isx} dx = \frac{\partial}{\partial t} \int_{-\infty}^{+\infty} u(x, t)e^{-2\pi isx} dx = \frac{\partial}{\partial t} \mathcal{F}u(s, t)
\]

Thus taking the Fourier transform (with respect to \( x \)) of both sides of the equation \( u_t(x, t) = u_{xx}(x, t) \) leads to

\[
\frac{\partial}{\partial t} \mathcal{F}u(s, t) = -4\pi^2 s^2 \mathcal{F}u(s, t)
\]

This is a differential equation in \( t \), an ordinary differential equation, despite the partial derivative symbol, and we can solve it:

\[
\mathcal{F}u(s, t) = \mathcal{F}u(s, 0) e^{-4\pi^2 s^2 t}
\]

What is the initial condition \( \mathcal{F}u(s, 0) \)?

\[
\mathcal{F}u(s, 0) = \int_{-\infty}^{+\infty} u(x, 0)e^{-2\pi isx} dx = \int_{-\infty}^{+\infty} f(x)e^{-2\pi isx} dx = \mathcal{F}f(s)
\]

Putting it all together

\[
\mathcal{F}u(s, t) = \mathcal{F}f(s) e^{-4\pi^2 s^2 t}
\]

We recognize that the exponential factor on the right hand side is the Fourier transform of the Gaussian

\[
g(x, t) = \frac{1}{\sqrt{4\pi t}} \exp \left( \frac{-x^2}{4t} \right)
\]

We then have a product of two Fourier transforms

\[
\mathcal{F}u(s, t) = \mathcal{F}f(s) \mathcal{F}g(s, t)
\]

and we invert this to obtain a convolution in the \( x \) domain \( u(x, t) = g(x, t) \ast f(x) \) or, written out

\[
u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} \exp \left( \frac{-(x-y)^2}{4t} \right) f(y) dy
\]

The function \( g(x, t) \) is called the heat kernel (or Green’s function, or fundamental solution) for the heat equation.