Exercise 1

a) Compute the derivative of the function \( f(x) = \log(2 + \sin(x^2)) \)

b) Compute the following integral \( \int_0^{\sqrt{\pi}} \frac{2x \cos(x^2)}{4 + \sin(x^2)} \, dx \)

Solution:

a) Using the rule to differentiate a composite function, \( f'(x) = \frac{2x \cos(x^2)}{2 + \sin(x^2)} \)

b) Notice that the function inside the integral is very similar to \( f'(x) \)... Therefore by the fundamental theorem of calculus, one obtains:

\[
\int_0^{\sqrt{\pi}} \frac{2x \cos(x^2)}{4 + \sin(x^2)} \, dx = \left[ \log(4 + \sin(x^2)) \right]_0^{\sqrt{\pi}} = \log(4 + \sin \pi) - \log(4 + \sin 0) = 0
\]

Exercise 2

Compute the following integral: \( \int_0^1 \frac{x^2}{\sqrt{x^3} + 1} \, dx \)

Solution: Setting \( \sqrt{x^3} + 1 = u \) you get \( \frac{du}{dx} = \frac{3}{2} \sqrt{x} \). In particular \( x^2 \, dx = (u - 1) \, du \). Therefore by substitution method you get

\[
\int_0^1 \frac{x^2}{\sqrt{x^3} + 1} \, dx = \frac{2}{3} \int_1^2 \frac{u - 1}{u} \, du = \left[ \frac{2}{3} (u - \log u) \right]_1^2 = \frac{2}{3} (1 - \log 2)
\]

Exercise 3

a) Compute the integral \( \int_{-1}^1 xe^{-x^2} \, dx \)

b) For any \( n \in \mathbb{N} \), can you say the value of \( \int_{-1}^1 x^{2n+1} e^{-x^2} \, dx \) without performing any computation? Justify your answer.

Solution:

a) Since the derivative of \( e^{-x^2} \) is \( -2x e^{-x^2} \), by the fundamental theorem of calculus one obtains

\[
\int_{-1}^1 x e^{-x^2} \, dx = \left[ \frac{e^{-x^2}}{-2} \right]_{-1}^1 = -\frac{1}{2} (e^{-1} - e^{-1}) = 0
\]

b) Since the function \( x^{2n+1} e^{-x^2} \) is odd and the interval \([-1, 1]\) is symmetric with respect to 0,

\[
\int_{-1}^1 x^{2n+1} e^{-x^2} \, dx = 0
\]
Exercise 4

Let $c$ be a real positive parameter. Consider the functions $f(x) = -\frac{x^4}{c} + 3c^3$ and $g(x) = 2cx^2$.

a) What are the abscissas $x_1 < x_2$ of the points where the graphs of $f$ intersects the graph of $g$?

b) Compute the area $A$ between the graphs of $f$ and $g$ on the interval $[x_1, x_2]$. What happens to this area when the parameter $c$ goes to $+\infty$?

Solution: In general, if $F$ is integrable and positive, $\int_a^b F(x) \, dx$ represents the area between the graph of $F$ and the $x$-axis, on the interval $[a,b]$.

a) The intersections are $x_1 = -c$ and $x_2 = c$. Indeed, by setting $t = x^2$,

$$f(x) = g(x) \iff -\frac{x^4}{c} + 3c^3 = 2cx^2 \iff t^2 + 2c^2 t - 3c^4 = 0 \iff t = c^2 \iff x = \pm c$$

(notice that we choose one solution of the quadratic equation in $t$ since $t = x^2$ have to be positive)

b) The area between the two graphs is the difference of the integrals:

$$A = \int_{x_1}^{x_2} f(x) \, dx - \int_{x_1}^{x_2} g(x) \, dx = \int_{-c}^{c} (-\frac{x^4}{c} + 3c^3 - 2cx^2) \, dx = \left[-\frac{x^5}{5c} - \frac{2c^3}{3} + 3c^3 x\right]_{-c}^c =$$

$$= \left(-\frac{c^4}{5} - \frac{2c^4}{3} + 3c^4\right) - \left(\frac{c^4}{5} + \frac{2c^4}{3} - 3c^4\right) = \frac{64}{15} c^4$$

Therefore $\lim_{c \to \infty} A(c) = \infty$. 

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