Advanced Position Control Schemes

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Summary

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In “nominal” conditions, it is possible to define control schemes:

- **decentralized (PID and variants)**
  - with a *cascade* configuration
  - with *feedforward* actions

- **centralized**
  - *inverse dynamics*  \( u = M(q)y + n(q, \dot{q}) \)
  - *PD with gravity compensation*  \( u = g(q) + K_P\ddot{q} - K_D\dot{q} \)

With these latter centralized control schemes, in principle, it is possible to achieve better performance from the robot, but they require a perfect knowledge of the dynamic model of the manipulator.
In particular, in these control schemes we assume:

- a *perfect modelling of all* the phenomena that affect the dynamics (friction, external forces, ...);
- a *perfect knowledge* of the parameters of the dynamic model of the robot;
- no external disturbances are applied to the robot (load, physical interactions, ...).

Very often, these assumptions are non realistic, and the presence of (large) modelling errors or of external disturbances may heavily deteriorate the achievable performance of the robots.

Therefore, it is necessary to adopt control techniques able to *compensate* these undesired phenomena.

- **Robust control**: aims at compensating directly possible errors deriving from model errors and/or external disturbances.
- **Adaptive control**: aims at modifying the control parameters in order to adapt to different working conditions, or at different values of the model parameters.
- **Learning control**: in case of repetitive tasks (quite common in industry), it may be useful to adopt control schemes able to learn cycle after cycle the most proper control input to reduce errors and optimize performances.
Part 1: Robust Control
Classical control techniques guarantee the achievement of some desired requirements in spite of external disturbances possibly applied to the controlled system, or changes in some of the parameters. Unfortunately, the range of these disturbances or variation of parameters is often quite limited.

Robust control techniques aim to directly compensate possible errors originated by:

- non precise knowledge of the kinematic and/or dynamic model (kinematic parameters, masses, load, friction, . . .)
- unknown external disturbances (variations of the load, applied forces, . . .)

These control techniques may deal with quite large uncertainties, but within a known range (maximum/minimum value).

Among the available methods, the Variable Structure control technique will be considered because of its relative simple implementation, and the possibility to be applied in many practical cases.
**Example.** Consider a second order dynamic system with two different "structures":

\[ \ddot{x} = -\psi x \quad \text{con} \quad \psi \in \{a_1^2, a_2^2\}, \quad (a_1^2 > 1 > a_2^2) \]

Each structure, considered alone, is simply stable:

\[
\begin{align*}
\psi &= a_1^2 \\
\psi &= a_2^2
\end{align*}
\]

Assuming that \(\psi\) is a control input, by using the commutation law:

\[
\psi = \begin{cases} 
a_1^2 & \text{if } \dot{x} \dot{x} > 0 \\
a_2^2 & \text{if } \dot{x} \dot{x} < 0
\end{cases}
\]

an asymptotically stable system is obtained.
Variable structure systems

**Example.** Consider the system

\[ \ddot{x} - \xi \dot{x} + \psi x = 0 \quad \xi > 0 \]

where \( \psi x \) can be considered as the ‘control’, and

\[ \psi = \begin{cases} 
-\frac{a}{\xi} & \text{(A)} \\
\frac{a}{\xi} & \text{(B)} 
\end{cases} \]

are two unstable structures \((a > 0)\).

In the case (A) there are two real eigenvalues, given by

\[ \lambda_{1,2} = \frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} + a} : \text{one is stable (sign ‘-’), and one unstable (sign ‘+’).} \]

In the case (B) there are complex conjugate eigenvalues with positive real part.
By choosing

$$\psi = \begin{cases} 
-a & \text{if } xs < 0 \\
 a & \text{if } xs > 0
\end{cases} \quad (A)$$

the resulting system is asymptotically stable.

$$s = -cx + \dot{x}$$

$$c = \frac{\xi}{2} - \sqrt{\frac{\xi^2}{4} + a} \quad (< 0)$$
Variable structure systems

The concept behind these examples may be generalized. Two different control laws are used:

\[ u_c = \begin{cases} 
K_1(x) & \text{if } S(x) > 0 \\
K_2(x) & \text{if } S(x) < 0 
\end{cases} \]

Simplified control law:

\[ u = \begin{cases} 
u^+ & \text{if } S(x) > 0 \\
u^- & \text{if } S(x) < 0 
\end{cases} \]

Sliding Surface: \( S(x) = 0 \)
Variable structure control

Let consider the system:

\[
\dot{x} = f(x, t, u) = f(x, t) + g(x, t)u(t) + \psi(t)
\]

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) the control action, and \(\psi(t)\) an external disturbance.

If there are variations of the parameters in \(f(x, t)\) and \(g(x, t)\), a simple linear controller is not able to guarantee the given control requirements, in particular when unknown disturbances are applied.

The basic idea of a \textit{variable structure controller} is to constrain the state \(x\) of the system to remain in a proper subregion \(S(x)\) of the state space: the \textit{sliding surface}.

Often, in case of more control inputs (\(m \geq 1\)) the surface \(S(x)\) is defined as the intersection of \(m\) surfaces \(S_i(x) = 0\).
The control law that allows to force $x$ on $S(x)$ is as follows:

$$u_i(x, t) = \begin{cases} 
  u_i^+(x, t) & \text{if } S_i(x) > 0 \\
  u_i^-(x, t) & \text{if } S_i(x) < 0 
\end{cases}$$

where $S_i(x)$ ($i = 1, \ldots, m$) are the sliding surfaces and $u_i^-(x, t)$ and $u_i^+(x, t)$ are two proper (constant) values (often the minimum and maximum).
Variable structure control

In a multi-dimensional case, the trajectories of the controlled system are:

\[ S_i(x) = 0 \]
\[ S_j(x) = 0 \]

Every time that the state \( x \) reaches a sliding surface \( S_i(x) \), it moves (slides) on it towards the intersecting space \( S(x) \).

If the region \( S(x) \) and the surfaces \( S_i(x) \) are chosen properly, after an initial transient period the state \( x \) of the system is constrained to remain in the final sliding region \( S(x) \), with no possibilities to “escape” even in presence of external disturbances.

Therefore, in this situation the dynamics of the controlled system is defined by \( S(x) \), that must be properly chosen in order to fast transient phases if disturbances or other undesired phenomena occur.
Variable structure control

In the multi-variable case, an alternative approach consists in computing the vector $z$, defining the distance from the sliding surface $S(x)$.

In this case, the control action is defined as

$$u = K \frac{z}{\|z\|}$$

The system state goes “directly” to the subspace $S(x)$, and not component by component as in the previous case. Once the state is in $S(x)$, it evolves in such a way that $S(x) = 0$. 

$$S(x) = 0$$

$$x_1$$

$$x_2$$

$$x_3$$

$$(x(0))$$
Variable structure control

One of the main features of the variable structure control (called also *sliding mode control*) is that the design of the controller consists in two steps:

1) **choice of the sliding region** $S(x)$ so that the controlled system has the desired dynamic behaviour (design requirements);

2) **choice of the control law** $u_i$ ($i = 1, \ldots, m$) in order to force the state on the sliding surface $S_i(x)$ even in case of parameters variations or external disturbances.
Example. Let consider the $2^{nd}$ order system

$$
\ddot{y} + a \dot{y} + b y = \psi(t) + u(t)
$$

where $u(t)$ is the control and $\psi(t)$ is an external disturbance, supposed to be bounded with bounded first order derivative

$$
|\psi(t)| < \Delta_0,
\quad |\dot{\psi}(t)| < \Delta_1
$$

Choice of the sliding surface (stability of the system):

$$
S(x) = \dot{y} + c y = 0
$$

with

$$
x = (y, \dot{y})
$$

When the state $x = (y, \dot{y}) \in S(x) = 0$:

- the dynamics of the controlled system is exponentially asymptotically stable

$$
\dot{y} = -c y \quad \implies \quad y(t) = y(0)e^{-ct}, \quad c > 0
$$

- the output of the system $y(t)$ goes to zero with a velocity that depends only on the parameter "$c$", that is on the chosen sliding surface

- the behaviour of the controlled system are does not dependent neither on the external disturbance or on changes in the parameters $a$ and $b$ (robustness).

Choice of the control: It is possible to proof that the discontinuous control law

$$
u(t) = -K \text{sgn} S(x)
$$

is able to force the state towards the sliding surface $S(x) = 0$. 
Variable structure control

Control \((y_d \neq 0)\)

\[ VS \quad G(s) \]

\[ y_d \quad \psi \quad u \quad y \]

Notice that if \(y_d \neq 0\), then \(x = (e, \dot{e})\), with \(e = y_d - y\).

We have total disturbance rejection (both external and due to parameters variations) if and only of:

\[ K > \Delta_0 > |\psi(t)|_{\text{max}} \] (2)
Let $a = 8$, $b = 15$, $c = 10$. The poles of the system are: 

\[
\begin{align*}
    p_1 &= -3 \\
    p_2 &= -5
\end{align*}
\]

Time evolution of the system with $u = 0$ for different initial conditions.

red lines: eigenvectors $(-3, -5)$;

green dashed line: sliding surface $S(x)$ $c = 10$)
Let $a = 8$, $b = 15$, $c = 10$, with $e = y_d - y$, $y_d = 2$, $x = (e, \dot{e})$.

**VS control without disturbance ($K = 150$)**
VS control with disturbance $\psi = -40$, $t = 1s$, $(K = 150)$
Variable structure control - Example

VS control with disturbance $\psi = -40$, $t = 1\text{s}$, $(K = 50)$
Variable structure control

**Continuous-time:**
When the state $x$ is on the sliding surface $S(x) = 0$, we have the *ideal sliding mode*:
- the control action $u(t)$ commutes with an *infinite frequency*,
- the oscillation produced on the output variable $y(t)$ has a *null amplitude*.

**Discrete-time:**
If the VS control law (1) is implemented in a discrete-time system (as it happens in practice) we have:
- a finite commutation frequency;
- a residual oscillation on the output variable $y(t)$ (chattering phenomenon).

The amplitude of the residual oscillation is proportional to the sampling period $T$ and to the value $K$ of the control action

$$|y(t)| \leq K \cdot T \quad (3)$$

Therefore, it is not possible to force the state “exactly” on the surface $S(x) = 0$ (ideal sliding mode), but rather is is possible to keep it within a *sufficiently small* neighborhood of it (discrete sliding mode).

From eq. (2) and (3) it is clear that if the disturbance $\psi(t)$ has a large amplitude, then also the value $K$ in the control law must be large. Therefore, also the chattering on $y(t)$ is large.
Variable structure control

**VS without disturbance** $K = 150$, $dt = 0.0001$ s

![Graphs showing position, velocity, S(e) values, and control action with $T = 0.0001$ s.](image)
Variable structure control

VS without disturbance $K = 150, dt = 0.001 s$
Variable structure control

VS without disturbance $K = 150, \; dt = 0.01 \; s$
The chattering problem

From a practical point of view, it is not possible to commute the control between the two values $u^+$, $u^-$ at an infinite frequency.

Therefore, “oscillations” are generated in the controlled system (or in the actuator), typically in a frequency range within the system’s bandwidth. Then, non modelled dynamics may be excited and undesired behaviours may be obtained.

Some methods have been proposed to avoid this problem. Among the most known (and easy to be implemented) we have:

- ‘Boundary layers’;
- DIC (Discrete Integral Control).
**BOUNDARY LAYERS:** a “region” with a proper amplitude is defined around the sliding surface; in this region, the control action is not discontinuous, but rather it changes in a continuous fashion, linearly proportional to the error:

$$\mathbf{x}(0)$$

$$S(x)$$

$$x_1$$

$$x_2$$

$$u^-$$

$$u^+$$

$$-\epsilon$$

$$\epsilon$$

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DIC (Discrete Integral Control).

Let assume that the disturbance is bounded with bounded first derivative:

\[ |\psi(t)| < \Delta_0, \quad |\dot{\psi}(t)| < \Delta_1 \]

In addition to the standard VS control term \((-k \text{ sgn } S(x))\), two additional contributions are used: a term proportional to the error \(-\lambda S(x)\), and one proportional to the integral if its sign

\[
\begin{align*}
    u(t) &= -\lambda S(x) - k \text{ sgn } S(x) - \dot{\psi}(t) \\
    \dot{\psi}(t) &= h \text{ sgn } S(x)
\end{align*}
\]

(4)

It is possible to proof the following:

**Property:** If the parameters \(k, h\) and \(\lambda\) of the control law (4) are chosen so that:

\[ k > 0, \quad h > \Delta_1, \quad h \lambda k > \Delta_1^2 (1 - \ln 2) \]

then the controlled system is globally asymptotically stable, and the state reaches the sliding surface \(S(x) = 0\) in a finite time.
Variable structure control

Comments:

- This property is valid only for continuous disturbances signals $\psi(t)$, with bounded derivative and acting in the controllable subspace (matching condition).

- If the proportional part ($-\lambda S(x)$) is not present, i.e. $\lambda = 0$, the control action is still able to stabilize the system, but only for “small” initial conditions.
Let consider again the system

\[ \ddot{y} + a \dot{y} + b y = \psi(t) + u(t) \]

with \( a = 8, \ b = 15, \ c = 10 \). Let \( x = y_d - y, \ y_d = 2 \).
Variable structure control - Example

VS+DIC without disturbance ($k = 4, \lambda = 20, h = 300$)
Variable structure control - Example

**VS with disturbance** $\psi = -40$, $t = 1s$, $(k = 4, \lambda = 20, h = 300)$
Robust control of industrial manipulators

In general, it is difficult to compensate exactly the dynamics of a robot manipulator. In practice, what we get is only a partial compensation. Let consider the inverse dynamics control:

\[ u = \hat{M}(q)y + \hat{n}(q, \dot{q}) \]  

(5)

where \( \hat{M} \) and \( \hat{n} \) represent the known part of the dynamic model. In general, uncertainties may be expressed as

\[ \tilde{M} = \hat{M} - M \]
\[ \tilde{n} = \hat{n} - n \]

By using (5), we have

\[ M\ddot{q} + n = \hat{M}y + \hat{n} \]

Since matrix \( M \) is invertible, we have

\[ \ddot{q} = y + (M^{-1}\hat{M} - I)y + M^{-1}\tilde{n} \]

\[ = y - \eta \]

with

\[ \eta = (I - M^{-1}\hat{M})y - M^{-1}\tilde{n} \]

N.B. If \( \hat{M} = M \) and \( \hat{n} = n \) \( \Rightarrow \) \( \eta = 0, \ \ddot{q} = y \).
By using the same control $y$ of the ideal case

$$y = \ddot{q}_d + K_D (\dot{q}_d - \dot{q}) + K_P (q_d - q)$$

The error dynamics $\ddot{q} = (q_d - q)$ is given by

$$\ddot{q} + K_D \dot{q} + K_P \ddot{q} = \eta$$

and therefore it is not possible to guarantee that $\ddot{q}$ goes to zero ($\eta$ depends on the uncertainties and on the nonlinear dynamics of the robot).
From

\[ \ddot{q} = y - \eta \]

we have

\[ \dddot{q} = \ddot{q}_d - y + \eta \]

Therefore, by defining \( \xi = [\ddot{q}^T, \dot{q}^T]^T \) as state vector, we obtain the first-order differential equation

\[ \dot{\xi} = H\xi + D(\ddot{q}_d - y + \eta) \]

with

\[ H = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \]

\[ D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{2n \times n} \]

The problem of tracking a given trajectory \((q_d, \dot{q}_d, \ddot{q}_d)\) may be solved by designing a control law \(y\) that stabilizes the error dynamics \(\xi\), that is nonlinear and time variant (because of the term \(\eta\)).
A consideration:
Because of the properties of any assigned trajectory (boundedness of the velocity/acceleration profiles) and considering the properties of the dynamic model of a robot manipulator (boundedness), it is possible to estimate the range of variability of the uncertainties $\eta$ affecting the error dynamics.

As a matter of fact, by analyzing each term we have

$$\sup_{t \geq 0} \| \dddot{q}_d \| < Q_M < \infty$$ \quad \forall \dddot{q}_d \quad (6)$$

$$\| I - M^{-1}(q) \hat{M}(q) \| \leq \alpha \leq 1$$ \quad \forall q \quad (7)$$

$$\| \bar{n} \| \leq N_M < \infty$$ \quad \forall q, \dot{q} \quad (8)$$
Robust control of industrial manipulators

- Eq. (6) considers the fact that any planned trajectory must have finite acceleration profiles.
- Eq. (7) is a consequence of the boundedness property of the inertia matrix $M$ (and therefore also of $M^{-1}$). Therefore

$$0 \leq M_m \leq \|M^{-1}(q)\| \leq M_M \leq \infty, \quad \forall q$$

Then, for example, we have

$$\|M^{-1}\hat{M} - I\| \leq \frac{M_M - M_m}{M_M + M_m} = \alpha \leq 1$$

in which $M^{-1} = M_M I$ and

$$\hat{M} = \frac{2}{M_M + M_m} I$$

have been considered. If $\hat{M}$ is chosen in a more accurate way, then the value of $\alpha$ is lower, up to the limit case of $\alpha = 0$ when $\hat{M} = M$.

- With respect to $\hat{n}$, we have to consider that $C(q, \dot{q})\dot{q}$ and $g(q)$ are always bounded.
Robust control of industrial manipulators

In order to compensate for the uncertainty term \( \eta \), we define a control law as

\[
y = \ddot{q}_d + K_D \dot{q} + K_P q + w
\]  

(9)

where

\[
\ddot{q}_d \quad \implies \quad \text{is a feedforward action}
\]

\[
K_D \dot{q} + K_P q \quad \implies \quad \text{stabilizes the error dynamics}
\]

\[
w \quad \implies \quad \text{compensates for uncertainties}
\]

The error dynamics \( \left( \dot{\xi} = H\xi + D(\ddot{q}_d - y + \eta) \right) \) in this case results

\[
\dot{\xi} = \tilde{H}\xi + D(\eta - w)
\]

where the eignevalues of matrix

\[
\tilde{H} = (H - DK) = \begin{bmatrix} O & I \\ -K_P & -K_D \end{bmatrix}
\]

have negative real part (\( K_P, K_D \) are positive definite matrices).
By properly choosing the two matrices $K_P$, $K_D$, for example as

$$
K_P = \text{diag}\{\omega_{n1}^2, \ldots, \omega_{nn}^2\} \quad K_D = \text{diag}\{2\delta_1\omega_{n1}, \ldots, 2\delta_n\omega_{nn}\}
$$

it is possible to obtain a desired (and decoupled) behaviour for the linear part of the error dynamics.

Notice that if $\eta = 0$, then by choosing $w = 0$ we obtain the previous control scheme.

Vice versa, if $\eta \neq 0$, a proper value for $w$ must be defined.

The Lyapunov method is adopted, choosing as candidate Lyapunov function

$$
V(\xi) = \xi^T Q \xi > 0 \quad \forall \xi \neq 0
$$

where $Q$ is a symmetric, positive definite matrix.
Candidate Lyapunov function:

$$V(\xi) = \xi^T Q \xi > 0 \quad \forall \xi \neq 0$$

By derivation, since $\dot{\xi} = \tilde{H} \xi + D(\eta - w)$, we have:

$$\dot{V} = \dot{\xi}^T Q \xi + \xi^T Q \dot{\xi}$$

$$= \xi^T (\tilde{H}^T Q + Q \tilde{H}) \xi + 2 \xi^T Q D(\eta - w)$$

$$= -\xi^T P \xi + 2 \xi^T Q D(\eta - w)$$

$$= -\xi^T P \xi + 2 z^T (\eta - w)$$

where

$$z = D^T Q \xi$$

and, since $\tilde{H} < 0$, then the equation

$$\tilde{H}^T Q + Q \tilde{H} = -P, \quad \forall \ P > 0$$

has a unique solution $Q$, symmetric and positive definite.

Notice that $\dot{V} < 0$ if

(a) $\xi \in \text{Null}(D^T Q)$

(b) $2 z^T (\eta - w) < 0$
By choosing the control law

\[ w = \frac{\rho}{\|z\|} z \quad \rho > 0 \]

we obtain

\[
\begin{align*}
    z^T (\eta - w) &= z^T \eta - \frac{\rho}{\|z\|} z^T z \\
    &\leq \|z\| \|\eta\| - \rho \|z\| \\
    &= \|z\| (\|\eta\| - \rho)
\end{align*}
\]

Therefore, for proper values of \( \rho \), i.e.

\[ \rho \geq \|\eta\| \quad \forall q, \dot{q}, \ddot{q}_d \]

we have \( \dot{V} < 0 \).
Robust control of industrial manipulators

As a matter of fact, since

$$\eta = (I - M^{-1}\hat{M})y - M^{-1}\tilde{n}$$

$$y = \ddot{q}_d + K_D \dot{q} + K_P \tilde{q} + w$$

and because of the boundedness of each term, we obtain

$$\|\eta\| \leq \|I - M^{-1}\hat{M}\| (\|\ddot{q}_d\| + \|K\| \|\xi\| + \|w\|) + \|M^{-1}\| \|\tilde{n}\|$$

$$\leq \alpha Q_M + \alpha \|K\| \|\xi\| + \alpha \rho + M_M N_M < \rho$$

from which

$$\rho \geq \frac{1}{1 - \alpha} (\alpha Q_M + \alpha \|K\| \|\xi\| + M_M N_M)$$

Therefore, with this choice of $w$ and $\rho$, we have that

$$\dot{V} = -\xi^T P \xi + 2z^T \left( \eta - \frac{\rho}{\|z\|} z \right) < 0 \quad \forall \xi \neq 0$$
Robust control of industrial manipulators

Overall control scheme.
Robust control of industrial manipulators

We have three types of contribution to the control action:

1. \( \hat{M}(q)y + \hat{n}(q, \dot{q}) \) guaranteeing a compensation, although approximated, of the nonlinear and coupling dynamic effects;

2. \( \ddot{q}_d + K_D \dot{q} + K_P q \) stabilizing the error dynamics; it may be interpreted as the combination of a feedforward \( (\ddot{q}_d + K_D \dot{q}_d + K_P q_d) \) and feedback \( (-K_D \dot{q} - K_P q) \) action.

3. \( w = \frac{\rho}{\|z\|}z \) guaranteeing robustness, since it compensates for uncertainties; the scalar term \( \rho > 0 \) is proportional to uncertainties.

A “vectorial” control action is obtained, since it is given by a vector of amplitude \( \rho \), directed as the unit vector \( \frac{z}{\|z\|} \), \( z = D^T Q \xi \).

The control action guarantees that in steady state \( \xi = \dot{\xi} = 0 \) (then \( \ddot{q} = \dot{q} = 0 \)). Moreover, the trajectories in the \( \xi \) space converge to the subspace defined by

\[
S(\xi) = z = D^T Q \xi = 0
\]
The *sliding* subspace \( S(\xi) = D^TQ\xi = 0 \) depends on the choice of the matrix \( Q \).

The chattering problem may be solved e.g. by the ‘boundary layers’ method:

\[
\mathbf{w} = \begin{cases} 
\frac{\rho}{\|\mathbf{z}\|}\mathbf{z} & \text{per } \|\mathbf{z}\| \geq \epsilon \\
\frac{\rho}{\epsilon}\mathbf{z} & \text{per } \|\mathbf{z}\| < \epsilon 
\end{cases}
\]
Part 2: Adaptive Control
Adaptive Control

In cases in which the uncertainties derive from a non perfect knowledge of the model’s parameters, it is possible to exploit *adaptive* control schemes able to:

- *estimate online (in real time)* the values of unknown parameters,
- *adapt the control parameters* on the basis of such estimation.

Block scheme of a (indirect) **Self Tuning Regulator (STR)**.

Block scheme of an implicit (direct) adaptive control scheme.
Control of the rotational velocity $\omega = \dot{\phi}$, where $\theta$ and $M$ are assumed non constant in time.

By defining:

- $\tau_m$: torque applied by the motor (joint $\phi$)
- $\tau_a$: torque necessary to compensate for friction
- $J$: inertia seen at the motor, $J = J(M, \theta)$

Balance of torques at the motor:

$$\frac{d}{dt} \left( J\ddot{\phi} \right) = J\dddot{\phi} + \frac{dJ}{d\theta} \dot{\theta} \ddot{\phi}, \quad \text{if } \dot{\theta} = 0 \implies J\dddot{\phi} = \tau_a + \tau_m$$

Let assume that typical variations of $J$, depending on the values of $\theta$ and $M$, may be in the range 1:50, reduced by the presence of the reduction gear to the range 1:5.
Adaptive Control - Example

The inertia $J$ at the motor is

$$J = J_a + \frac{J_m(M, \theta)}{k_r^2}$$

where: $k_r$ reduction ration, $J_a$ motor inertia, $J_m$ load inertia (variable in time).

If $\theta$ is constant and $\tau_a = 0$, the balance of the torques gives

$$J\ddot{\phi} = \tau_m$$

The torque $\tau_m$ is proportional to the motor current $i$

$$\tau_m = k_m i$$

Control scheme:
By adopting a PI controller, the current is computed as

\[ i = k \left[ (\omega_r - \omega) + \frac{1}{T_i} \int_0^t (\omega_r - \omega) dt \right] \]

Therefore

\[ J \frac{d^2 \omega}{dt^2} + k_m k \frac{d \omega}{dt} + \frac{k_m k}{T_i} \omega = k_m k \frac{d \omega_r}{dt} + \frac{k_m k}{T_i} \omega_r \]

Given some desired values \( \delta_0, \omega_0 \) for the damping coefficient and natural frequency, by choosing

\[ k = \frac{2\delta_0 \omega_0 J}{k_m}, \quad T_i = \frac{2\delta_0}{\omega_0} \]

we have

\[ \frac{\omega(s)}{\omega_r(s)} = G_0(s) = \frac{2\delta_0 \omega_0 s + \omega_0^2}{s^2 + 2\delta_0 \omega_0 s + \omega_0^2} \]
If the parameters of the controller are computed for $J_0$ (nominal value), while $J \neq J_0$, we have

$$G'_0(s) = \frac{2\delta_0\omega_0 s J_0 / J + \omega_0^2 J_0 / J}{s^2 + 2\delta_0 \omega_0 s J_0 / J + \omega_0^2 J_0 / J}$$

$\implies \omega_n = \omega_0 \sqrt{J_0 / J}, \quad \delta = \delta_0 \sqrt{J_0 / J}$

If $\omega_0 = 1 \text{ rad/s}$, $\delta_0 = 0.7$:

- when $J = 2J_0$ $\implies \omega_n = 0.7071$, $\delta \approx 0.5$
- when $J = \frac{1}{2}J_0$ $\implies \omega_n = 1.4142$, $\delta \approx 1$
Adaptive Control - Example

Variation of $\delta$ and $\omega_n$ when $J/J_0 \in [0.2 \rightarrow 20]$

- Variation of the inertia $J_m = M(L \cdot \sin \theta)^2$ (load side) for $\theta \in [0 \rightarrow \pi/2]$ and $M \in [5 \rightarrow 12.5]$ ($L = 2$)

- System output $\omega$ (red: ideal behaviour) for different values of $J \in [0.2 \rightarrow 5]J_0$
Adaptive Control

In robotics, in order to design an adaptive control law, we exploit the property of the dynamic model to be linear in the parameters:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = Y(q, \dot{q}, \ddot{q}) \alpha = u \]

where \( \alpha \) is a \((p \times 1)\) vector of constant parameters and \( Y \) is an \((n \times p)\) matrix, function of joint positions, velocities and accelerations.

Let consider the control law:

\[ u = M(q)\dot{q}_r + C(q, \dot{q})\dot{q}_r + D\dot{q}_r + g(q) + K_D\sigma \quad (10) \]

where

\[ \dot{q}_r = \dot{q}_d + \Lambda\dot{q} \]
\[ \ddot{q}_r = \ddot{q}_d + \Lambda\ddot{q} \]

and \( K_D \) a positive definite matrix. The term \( K_D\sigma \) is equivalent to a PD control action on the tracking error assuming

\[ \sigma = \dot{q}_r - \dot{q} = \ddot{q} + \Lambda\ddot{q} \quad \quad [K_P = \Lambda K_D] \]

The overall dynamics is then described by

\[ M(q)\ddot{\sigma} + C(q, \dot{q})\sigma + D\sigma + K_D\sigma = 0 \]
The stability of the system may be proven with the Lyapunov method. As a matter of fact, let choose the candidate Lyapunov function

\[ V(\sigma, \tilde{q}) = \frac{1}{2} \sigma^T M(q) \sigma + \frac{1}{2} \tilde{q}^T \tilde{A} \tilde{q} > 0 \quad \forall \sigma, \tilde{q} \neq 0 \]

where \( A \) is a positive definite matrix. The time derivative of \( V \) is

\[
\dot{V} = \sigma^T M(q) \dot{\sigma} + \frac{1}{2} \sigma^T \dot{M}(q) \sigma + \tilde{q}^T \dot{A} \tilde{q}
\]

\[
= \sigma^T M \left[ -M^{-1} C \sigma - M^{-1} D \sigma - M^{-1} K_D \sigma \right] + \frac{1}{2} \sigma^T \dot{M} \sigma + \tilde{q}^T \dot{A} \tilde{q}
\]

\[
= \frac{1}{2} \sigma^T \left( \dot{M} - 2C \right) \sigma - \sigma^T (D + K_D) \sigma + \tilde{q}^T \dot{A} \tilde{q}
\]

\[
= -\sigma^T (D + K_D) \sigma + \tilde{q}^T \dot{A} \tilde{q}
\]

\[
= -\sigma^T D \sigma - (\dot{\tilde{q}} + \Lambda \tilde{q})^T K_D (\dot{\tilde{q}} + \Lambda \tilde{q}) + \tilde{q}^T A \dot{\tilde{q}}
\]

\[
A = 2\Lambda K_D
\]

Therefore \( \dot{V} < 0 \), and \( \dot{V} = 0 \) only when \( \tilde{q} = \dot{\tilde{q}} \equiv 0 \)

\[ [\tilde{q}^T, \sigma^T]^T = 0 \text{ is globally asymptotically stable.} \]

The system evolves on \( \sigma = 0 \) without a high-frequency control action.
On the basis of the previous result, let consider the control law

\[ u = \hat{M}(q)\ddot{q}_r + \hat{C}(q, \dot{q})\dot{q}_r + \hat{D}\dot{q}_r + \hat{g}(q) + K_D\sigma \]

\[ = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\hat{\alpha} + K_D\sigma \]

computed using the estimation \( \hat{\alpha} \) of the parameters of the dynamic model.

Notice that \( Y \) does not depend on \( \ddot{q} \) (from the definition of \( q_r, \dot{q}_r \)). With this control, the overall dynamics is described by:

\[ M(q)\ddot{\sigma} + C(q, \dot{q})\dot{\sigma} + D\dot{\sigma} + K_D\sigma = -\tilde{M}(q)\ddot{q}_r - \tilde{C}(q, \dot{q})\dot{q}_r - \tilde{D}\dot{q}_r - \tilde{g}(q) \]

\[ = -Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\tilde{\alpha} \]

where

\[ \tilde{M} = \hat{M} - M, \quad \tilde{C} = \hat{C} - C, \quad \tilde{D} = \hat{D} - D, \quad \tilde{g} = \hat{g} - g, \quad \tilde{\alpha} = \hat{\alpha} - \alpha \]
In order to prove the stability of the system with the new control law, consider now a candidate Lyapunov function defined as

\[
V(\sigma, \tilde{q}, \tilde{\alpha}) = \frac{1}{2} \sigma^T M(q) \sigma + \tilde{q}^T \Lambda K_D \tilde{q} + \frac{1}{2} \tilde{\alpha}^T K_\alpha \tilde{\alpha} > 0 \quad \forall \sigma, \tilde{q}, \tilde{\alpha} \neq 0
\]

where \(K_\alpha\) is a symmetric positive definite matrix. The time derivative of \(V\) is now:

\[
\dot{V} = -\sigma^T D \sigma - \tilde{q}^T K_D \tilde{q} - \tilde{q}^T \Lambda K_D \Lambda \tilde{q} + \tilde{\alpha}^T (K_\alpha \tilde{\alpha} - Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma)
\]

as in the previous case due to uncertainties.
Adaptive Control

The function $\dot{V}$ is negative if the parameters are “adapted” according to
$$\dot{\alpha} = \hat{\alpha} - \alpha \implies \hat{\alpha} = \hat{\alpha}$$

$$\dot{\hat{\alpha}} = K^{-1}_\alpha Y^T (q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma$$

As a matter of fact, we have
$$\dot{V} = -\sigma^T D\sigma - \dot{q}^T K_D \dot{q} - \ddot{q}^T \Lambda K_D \Lambda \ddot{q} < 0$$

Therefore, with the control law
$$u = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \hat{\alpha} + K_d (\dot{\hat{q}} + \Lambda \hat{q})$$

where the parameters are computed according to
$$\dot{\hat{\alpha}} = K^{-1}_\alpha Y^T (q, \dot{q}, \dot{q}_r, \ddot{q}_r) \sigma$$

the overall dynamics converges to $\sigma = 0, \dot{\hat{q}} = 0$ and then $\hat{q}, \dot{\hat{q}}$ converge to zero and moreover $\hat{\alpha}$ is bounded.

Notice that from
$$M(q) \dot{\sigma} + C(q, \dot{q}) \sigma + D\sigma + K_D \sigma = -Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) \hat{\alpha}$$

in steady state we have
$$Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) (\hat{\alpha} - \alpha) = 0 \iff \hat{\alpha} \to \alpha$$
Adaptive Control

The overall control scheme is

\[
\dot{q}_d \quad \dot{q}_r \quad \Lambda \quad \ddot{q} \quad K_{\alpha}^{-1} Y T \quad \dot{\hat{\alpha}} \quad \int \dot{\hat{\alpha}} \quad Y \quad u \quad \text{Manip.} \quad \dot{q} \quad q
\]

This is an *implicit* adaptive control. There are three contributions:

- term \( Y \hat{\alpha} \), riconducibile ad una azione a dinamica inversa;
- term \( K_D \sigma \), azione stabilizzante di tipo PD sull’errore;
- estimated parameter vector \( \hat{\alpha} \), aggiornato secondo una tecnica *a gradiente*, la matrice \( K_\alpha \) determina la velocità di convergenza della stima dei parametri.
Trajectories and estimation problems

Tracking error \[\implies\] asymptotically stable \[\sigma \to 0\]

Estimation error \[\implies\] bounded

In general, we have to guarantee some conditions on the signals used for the parameters estimation: *persistently exciting signals*.

These conditions affect the properties of the *regression matrix* (in this case the matrix \(Y\)).

Notice that \(Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) = Y(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d)\).

In brief, a relationship such as

\[
a I_p \leq \int_{t_0}^{t_0+T} Y^T(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d)Y(q, \dot{q}, q_d, \dot{q}_d, \ddot{q}_d)dt \leq b I_p \quad \forall t_0, \ T, \ a, b > 0
\]

must be verified. Since the error dynamics is asymptotically stable, the previous relationship may be rewritten as

\[
a I_p \leq \int_{t_0}^{t_0+T} Y^T(q_d, \dot{q}_d, \ddot{q}_d)Y(q_d, \dot{q}_d, \ddot{q}_d)dt \leq b I_p \quad \forall t_0, \ T, \ a, b > 0
\]

Therefore, the trajectories applied to the manipulator must guarantee that matrix \(Y\) is “sufficiently rich” in the considered time interval, in order to “fill” all the \(p\)-dimensional parameters space. In this manner, a proper estimation of all the parameters may be obtained.
In conclusion:

**Adaptive control:**
- does not compensate directly external disturbances,
- depends on the considered model (structure of the model, number of parameters, ...),
- tries to compensate the effects of disturbances by modifying the parameters (does not act directly *against* the disturbances),
- ‘smooth’ control action.

**Robust control:**
- direct compensation of disturbances,
- chattering phenomenon.
Part 3:

Advanced Control Schemes - Examples
Some simulation results obtained with the previous control schemes are now reported. In particular, the results refer to the use of:

- Adaptive control
- VS control
- PD control + gravity compensation
- Inverse dynamics control

both in the “nominal” case and with external disturbances and/or parametric uncertainties.

A planar 2 dof robot is considered, and the overall control scheme is the following:
The desired trajectory is defined in the work-space, as a fifth-order polynomial function. Its time duration is $\Delta T = 2s$, with $x_i = [2, 0]^T$, and $x_f = [-1, 1.5]^T$.
Adaptive Control

Linearity of the dynamic model

\[ Y(q, \dot{q}, \ddot{q}) \alpha = \tau \]

\[
\begin{align*}
[m_1 a_{C1}^2 + m_2 (a_1^2 + a_{C2}^2 + 2a_1a_{C2}C_2) + \tilde{l}_1 + \tilde{l}_2] \dot{\theta}_1 &+ [m_2 (a_{C2}^2 + a_1a_{C2}C_2) + \tilde{l}_2] \dot{\theta}_2 \\
&- m_2 a_1 a_{C2} S_2 \dot{\theta}_2^2 - 2m_2 a_1 a_{C2} S_2 \dot{\theta}_1 \dot{\theta}_2 \\
&+ (m_1 a_{C1} + m_2 a_1) g C_1 + m_2 g a_{C2} C_{12} = \tau_1 \\

[m_2 (a_{C2}^2 + a_1 a_{C2} C_2) + \tilde{l}_2] \ddot{\theta}_1 &+ [m_2 a_{C2}^2 + \tilde{l}_2] \ddot{\theta}_2 \\
&+ m_2 a_1 a_{C2} S_2 \dot{\theta}_1^2 \\
&+ m_2 g a_{C2} C_{12} = \tau_2
\end{align*}
\]

By inspection, it is possible to define the parameters vector

\[ \alpha = [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5]^T \]

with

\[ \alpha_1 = m_1 a_{C1}, \quad \alpha_2 = m_1 a_{C1}^2 + \tilde{l}_1, \quad \alpha_3 = m_2, \quad \alpha_4 = m_2 a_{C2}, \quad \alpha_5 = m_2 a_{C2}^2 + \tilde{l}_2 \]
Adaptive Control

Moreover:

\[ Y(q, \dot{q}, \ddot{q}) = \begin{bmatrix} y_{11} & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & 0 & y_{24} & y_{25} \end{bmatrix} \]

with

\[
\begin{align*}
y_{11} &= gC_1 \\
y_{12} &= \ddot{\theta}_1 \\
y_{13} &= a_1^2 \ddot{\theta}_1 + a_1 gC_1 \\
y_{14} &= 2a_1 C_2 \ddot{\theta}_1 + a_1 C_2 \ddot{\theta}_2 - 2a_1 S_2 \dot{\theta}_1 \dot{\theta}_2 - a_1 S_2 \dot{\theta}_1^2 + gC_{12} \\
y_{15} &= \ddot{\theta}_1 + \ddot{\theta}_2 \\
y_{24} &= a_1 C_2 \ddot{\theta}_1 + a_1 S_2 \dot{\theta}_1^2 + gC_{12} \\
y_{25} &= \dot{\theta}_1 + \dot{\theta}_2
\end{align*}
\]

Notice that the terms \( y_{ij} \) depend on \( q, \dot{q}, \ddot{q}, g \) and on \( a_1 \).
$T_s = 0.002 \text{ s}$

Real values of the parameters:

- $\alpha_1 = m_1 a_1 c_1 = 25$
- $\alpha_2 = m_1 a_1^2 + \tilde{l}_1 = 22.5$
- $\alpha_3 = m_2 = 50$
- $\alpha_4 = m_2 a_2 c_2 = 25$
- $\alpha_5 = m_2 a_2^2 + \tilde{l}_2 = 22.5$

Initial values: Final values:

- $\hat{\alpha}_1 = 22.5$ $\hat{\alpha}_1 = 26.9997$
- $\hat{\alpha}_2 = 20.25$ $\hat{\alpha}_2 = 21.4363$
- $\hat{\alpha}_3 = 45$ $\hat{\alpha}_3 = 50.6860$
- $\hat{\alpha}_4 = 22.5$ $\hat{\alpha}_4 = 24.7509$
- $\hat{\alpha}_5 = 20.25$ $\hat{\alpha}_5 = 17.8884$
Adaptive Control

- Norma dell’errore cartesiano; ex (dot), ey (dash)
  - Time [s]
  - [m]

- Norma dell’errore di giunto; eq1 (dot), eq2 (dash)
  - Time [s]
  - [deg]

- Andamento della stima dei parametri
  - Time [s]
  - [°C]

- Andamento delle coppie
  - Time [s]
  - [N·m]
With disturbance torques $\tau_{\text{dis}} = [80, \ 80]^T$, $t = 3s$, $T_s = 0.002$ s:

**True parameters:**

\[
\begin{align*}
\alpha_1 &= m_1 a C_1 = 25 \\
\alpha_2 &= m_1 a^2 C_1 + \tilde{I}_1 = 22.5 \\
\alpha_3 &= m_2 = 50 \\
\alpha_4 &= m_2 a C_2 = 25 \\
\alpha_5 &= m_2 a^2 C_2 + \tilde{I}_2 = 22.5
\end{align*}
\]

**Initial values:**

\[
\begin{align*}
\hat{\alpha}_1 &= 22.5 \\
\hat{\alpha}_2 &= 20.25 \\
\hat{\alpha}_3 &= 45 \\
\hat{\alpha}_4 &= 22.5 \\
\hat{\alpha}_5 &= 20.25
\end{align*}
\]

**Final values:**

\[
\begin{align*}
\hat{\alpha}_1 &= 27.7443 \\
\hat{\alpha}_2 &= 21.4479 \\
\hat{\alpha}_3 &= 51.4422 \\
\hat{\alpha}_4 &= 33.7187 \\
\hat{\alpha}_5 &= 17.9385
\end{align*}
\]
Adaptive Control

Norma dell’errore cartesiano; ex (dot), ey (dash)

Norma dell’errore di giunto; eq1 (dot), eq2 (dash)

Andamento della stima dei parametri

Andamento delle coppie
VS Control

\[
e = \begin{bmatrix} q_{d,1} \\ q_{d,2} \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

\[
S(x) = \dot{e} + Ce,
\]

\[
C = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}
\]

\[K = 2000\]

\[T_s = 0.001 \text{ s}\]
VS Control

Norma dell'errore cartesiano; ex (dot), ey (dash)

Norma dell'errore di giunto; eq1 (dot), eq2 (dash)
With disturbance torques $\tau_{dis} = [80, 80]^T$, $t = 3s$:

$$e = \begin{bmatrix} q_{d,1} \\ q_{d,2} \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$S(x) = \dot{e} + Ce,$$

$$C = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$K = 2000$$

$$T_s = 0.001 \text{ s}$$
VS Control

- Norma dell'errore cartesiano: ex (dot), ey (dash)
- Norma dell'errore di giunto: eq1 (dot), eq2 (dash)
PD control + gravity compensation

\[ e = \begin{bmatrix} q_{d,1} \\ q_{d,2} \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \]

\[ \tau = K_p e - K_d \dot{q} + g(q) \]
PD control + gravity compensation

- Norma dell'errore cartesiano: ex (dot), ey (dash)
- Norma dell'errore di giunto; eq1 (dot), eq2 (dash)

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PD control + gravity compensation

With disturbance torques $\tau_{\text{dis}} = [80, \ 80]^T$, $t = 3s$:

$$e = \begin{bmatrix} q_{d,1} \\ q_{d,2} \end{bmatrix} - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\tau = K_p e - K_d \dot{q} + g(q)$$

final (steady state) error $\neq 0$
PD control + gravity compensation

**Graphs:**
- **Norma dell'errore cartesiano:** $ex$ (dot), $ey$ (dash)
- **Norma dell'errore di giunto:** $eq1$ (dot), $eq2$ (dash)
- **Taus:** $\tau_{a1}$, $\tau_{a2}$

**Figures:**
- Time [s] vs. Norm [m] for $ex$, $ey$, $eq1$, $eq2$
- Time [s] vs. $T_{a1}$, $T_{a2}$

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Inverse dynamics (nominal case)

\[ y = \ddot{q}_d + K_d \dot{e} + K_p e \]

\[ \tau = My + C\dot{q} + g(q) \]
Inverse dynamics control (nominal case)
Inverse dynamics control (non nominal conditions)

Real values of the parameters:
\[
\begin{align*}
\alpha_1 &= m_1 a_{C1} = 25 \\
\alpha_2 &= m_1 a_{C1}^2 + l_1 = 22.5 \\
\alpha_3 &= m_2 = 50 \\
\alpha_4 &= m_2 a_{C2} = 25 \\
\alpha_5 &= m_2 a_{C2}^2 + l_2 = 22.5 \\
\end{align*}
\]

Values used in the control:
\[
\begin{align*}
\hat{\alpha}_1 &= 22.5 \\
\hat{\alpha}_2 &= 20.25 \\
\hat{\alpha}_3 &= 45 \\
\hat{\alpha}_4 &= 22.5 \\
\hat{\alpha}_5 &= 20.25 \\
\end{align*}
\]

\[
y = \ddot{q}_d + K_d \dot{e} + K_p e
\]

\[
\tau = \ddot{M}y + \hat{C}q + \hat{g}(q)
\]
Inverse dynamics control (non nominal conditions)