The stratified coastal ocean

Part 1
Main reference

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Circulation in the Coastal Ocean. Chap. 3: The behaviour of the Stratified Sea. Sections. 3.1, 3.2, 3.6
Focus on state of motion of a stratified coastal ocean of constant depth.

Motion depending on the “perturbation” of the “state of rest” relative to the rotating Earth.

Equilibrium density distribution $\rho(z)$.

(might contain 1 or more “mixed” layers of density $\rho$)

Small vertical displacement of water particles $\rightarrow$ local density perturbations $\rho'$

$$\rho = \rho_0 + \rho'$$

$$\rho' < \rho_0$$

Local density perturbations $\rightarrow$ pressure perturbations via hydrostatic equation.

$$\frac{\partial p}{\partial z} = -\rho g$$

$$\frac{\partial p}{\partial z} = -(\rho_0 + \rho') g$$

$$p = (p_0 + p')$$

$$\frac{\partial p_0}{\partial z} = -\rho_0 g$$

$$\frac{\partial p'}{\partial z} = -\rho' g$$
The appropriate linearised equations are (under hydrostatic approximation) are:

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v &= -\frac{1}{\rho} \frac{\partial p'}{\partial x} + \frac{A_v}{\rho} \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial v}{\partial t} + f u &= -\frac{1}{\rho} \frac{\partial p'}{\partial y} + \frac{A_v}{\rho} \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial p'}{\partial z} &= -g \rho' \\
\frac{\partial \rho'}{\partial t} + w \frac{d \rho}{dz} &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]
The appropriate linearised equations are (under hydrostatic approximation) are:

\[
\frac{\partial \rho'}{\partial t} + w \frac{d \rho}{dz} = 0
\]

Density conservation

\( \rho' \) and \( w \): small (perturbation) quantities

\( \frac{d \rho}{dz} \): Vertical density gradient in the undisturbed mean state (can be large)

The (small) vertical displacement of particles is defined as:

\[
\eta = \int_{0}^{t} w \, dt
\]

N.B.: Here \( \eta (x,y,z,t) \) refers to the particles at all levels and NOT Only at the surface
An equation connecting $p'$ to $\eta$ can be found starting from

$$\frac{\partial p'}{\partial z} = -g \rho'$$

and

$$\frac{\partial \rho'}{\partial t} + w \frac{d\rho}{dz} = 0$$

Posing $wt = \eta$

$$\frac{\partial p'}{\partial z} = g\eta \frac{\partial \rho}{\partial z}$$
The surface boundary condition is (assuming constant atmospheric pressure):

\[
\frac{\partial p'}{\partial z} = g \eta \frac{\partial \rho}{\partial z}
\]

Integration of the equation connecting \(p'\) to \(\eta\) above yields

\[
p' = g \left( \rho \bigg|_{z=\eta} \eta_s - \int_{z}^{0} \frac{\partial p'}{\partial z'} \eta(z')dz' \right)
\]

The bottom boundary condition is

\[
w_{z=-H}=0 \quad (H=constant)
\]
The solution of

\[
\begin{align*}
\frac{\partial u}{\partial t} - f v &= - \frac{1}{\rho} \frac{\partial p'}{\partial x} + \frac{A_v}{\rho} \frac{\partial^2 u}{\partial z^2} \\
\frac{\partial v}{\partial t} + f u &= - \frac{1}{\rho} \frac{\partial p'}{\partial y} + \frac{A_v}{\rho} \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial p'}{\partial z} &= -g \rho' \\
\frac{\partial \rho'}{\partial t} + w \frac{d \rho}{d z} &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{align*}
\]

can be expressed by a combination of modes, each one characterised by a certain vertical distribution of Horizontal velocities and vertical displacement:

\[
\begin{align*}
u &= \sum_k U_k(x,y,t) \frac{dQ_k}{dz} \\
v &= \sum_k V_k(x,y,t) \frac{dQ_k}{dz} \\
\eta &= \sum_k Z_k(x,y,t)Q_k
\end{align*}
\]

\((k = 1, 2, 3, \ldots)\)

Distribution function \(Q_k\) is non-dimensional, therefore \(U_k, V_k\) have the physical dimensions of transport (velocity X Length)
Two layer model of the stratified Ocean

Idealised representation of the stratified coastal Ocean:
A sharp pycnocline separating two fluids of slightly different density.

Sharp variation of density (12 density units in 15 m) due to thermocline only (fresh water case)

Therefore an abrupt density discontinuity is not an unrealistic case.
Two layer model of the stratified Ocean

η': free surface elevation

ρ: upper layer (of “thickness” \( h \)) density

ρ': lower layer (of “thickness” \( h' \)) density

\( H = h + h' \)
Two layer model of the stratified Ocean

Main assumption both surface ($\eta$) and interface ($\eta'$) displacements are small with respect to the equilibrium depth of either layer ($h$ or $h'$).

Interface and bottom friction are assumed to be zero.

The proportionate density defect of the top layer:

$$\varepsilon = \frac{\rho' - \rho}{\rho'} = 1 - \frac{\rho}{\rho'}$$

Is in the order of $10^{-3}$.

Transport components (depth integrated velocities) are defined separately for each layer:

**Top layer**

$$(U, V) = \int_{-h+\eta'}^{\eta} (u, v) \, dz$$

**Lower layer**

$$(U', V') = \int_{-h}^{-h+\eta'} (u, v) \, dz$$

Pressure in the two layers is defined by the hydrostatic approximation:

$$p = \rho g (\eta - z)$$

$$p' = g \left[ \rho (\eta - \eta' + h) + \rho' (\eta' - h - z) \right]$$
Two layer model of the stratified Ocean

Depth integration of the linearised equation of motion and continuity lead to the following transport equations:

**Top layer**

\[
\frac{\partial U}{\partial t} - fV = -gh \frac{\partial \eta}{\partial x} + \frac{\tau^{(x)}}{\rho_0}
\]

\[
\frac{\partial V}{\partial t} + fU = -gh \frac{\partial \eta}{\partial y} + \frac{\tau^{(y)}}{\rho_0}
\]

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial}{\partial t} (\eta - \eta')
\]

**Lower layer**

\[
\frac{\partial U'}{\partial t} - fV' = -gh' \left( \frac{\rho}{\rho'} \frac{\partial \eta'}{\partial x} + \varepsilon \frac{\partial \eta'}{\partial x} \right)
\]

\[
\frac{\partial V'}{\partial t} + fU' = -gh' \left( \frac{\rho}{\rho'} \frac{\partial \eta'}{\partial y} + \varepsilon \frac{\partial \eta'}{\partial x} \right)
\]

\[
\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \eta'}{\partial t}
\]
Two layer model of the stratified Ocean

The pressure term on the lower layer

Recall that

\[ p' = g \left[ \rho(\eta - \eta' + h) + \rho'(\eta' - h - z) \right] \] and \( \varepsilon = 1 - \frac{\rho}{\rho'} \)

Then

\[ \frac{1}{\rho'} \frac{\partial p'}{\partial x} = \frac{1}{\rho'} \frac{\partial}{\partial x} g \left[ \rho(\eta - \eta' + h) + \rho'(\eta' - h - z) \right] = \]

\[ = \frac{\rho}{\rho'} g \frac{\partial}{\partial x} (\eta - \eta') + g \frac{\partial}{\partial x} \eta' = \]

\[ \frac{\rho}{\rho'} g \frac{\partial \eta}{\partial x} + g \frac{\partial \eta'}{\partial x} \left( 1 - \frac{\rho}{\rho'} \right) = \]

\[ = g \left( \frac{\rho}{\rho'} \frac{\partial \eta}{\partial x} + \varepsilon \frac{\partial \eta'}{\partial x} \right) \] upon vertical integration
Two layer model of the stratified Ocean

Top layer

Lower layer

Equations for top and bottom layers are coupled, \( \eta' \) appearing in the 1\textsuperscript{st} equation set and \( \eta \) in the 2\textsuperscript{nd}

A linear combination of the 2 sets of 3 equations gives rise to the following equation of motion and continuity:

\[
\begin{align*}
\frac{\partial}{\partial t} (aU + bU') - f(aV + bV') &= -gh \frac{\partial}{\partial x} \left( a\eta + b \frac{h'}{h} \rho' \eta + b \frac{h'}{h} \epsilon \eta' \right) \\
\frac{\partial}{\partial t} (aV + bV') + f(aU + bU') &= -gh \frac{\partial}{\partial y} \left( a\eta + b \frac{h'}{h} \rho' \eta + b \frac{h'}{h} \epsilon \eta' \right) \\
\frac{\partial}{\partial x} (aU + bU') + \frac{\partial}{\partial y} (aV + bV') &= -\frac{\partial}{\partial t} (a\eta - a\eta' - b\eta')
\end{align*}
\]
Two layer model of the stratified Ocean

\[
\begin{align*}
\frac{\partial}{\partial t}(a U + b U') - f (a V + b V') &= -gh \frac{\partial}{\partial x} \left( a \eta + b \frac{h'}{h \rho'} \eta + b \frac{h'}{h} \varepsilon \eta' \right) \\
\frac{\partial}{\partial t}(a V + b V') + f (a U + b U') &= -gh \frac{\partial}{\partial y} \left( a \eta + b \frac{h'}{h \rho'} \eta + b \frac{h'}{h} \varepsilon \eta' \right) \\
\frac{\partial}{\partial x}(a U + b U') + \frac{\partial}{\partial y}(a V + b V') &= -\frac{\partial}{\partial t}(a \eta - a \eta' - b \eta')
\end{align*}
\]

*a and b are two arbitrary constants’

A set of equations for the linear transport combinations will be of the form of the homogeneous transport equations:

\[
\begin{align*}
\frac{\partial U}{\partial t} - f V &= -gH \frac{\partial \eta}{\partial x} + \frac{1}{\rho_0} \left( \tau_w^x - \tau_b^x \right) \frac{\partial V}{\partial t} + f U = -gH \frac{\partial \eta}{\partial y} + \frac{1}{\rho_0} \left( \tau_w^y - \tau_b^y \right) \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial \eta}{\partial t}
\end{align*}
\]

If the pressure variables in the equation of motion and in the equation of continuity (right hand side of both) are the same except for an arbitrary constant \(\beta\):

\[
\beta(a \eta - a \eta' - b \eta') = a \eta + b \frac{h'}{h \rho'} \eta + b \frac{h'}{h} \varepsilon \eta'
\]
Two layer model of the stratified Ocean

\[ \beta \left( a\eta - a\eta' - b\eta' \right) = a\eta + b \frac{h' \rho}{h \rho'} \eta + b \frac{h'}{h} \varepsilon \eta' \]

Coefficients for both \( \eta \) and \( \eta' \) have to agree separately:

\[ \beta a = a + b \frac{h' \rho}{h \rho'} \]  
\[ -\beta a + \beta b = \frac{h'}{h} \varepsilon b \]

The two equations for \( a \) and \( b \) are compatible only if the determinant vanishes that is to say:

\[ \beta^2 - \beta \left( \frac{h'}{h} + 1 \right) + \frac{h'}{h} \varepsilon = 0 \]

The two roots of the above equation can be expanded (for the small \( \varepsilon \) values) as

\[ \beta_1 = 1 + \frac{h'}{h} - \varepsilon \frac{h'}{h + h'} \]
\[ \beta_2 = \varepsilon \frac{h'}{h + h'} + O(\varepsilon^2) \]
The ratio of the 2 constants $a$ and $b$ now can be determined according to:

$$\frac{a}{b} = 1 - \frac{h'}{h\beta}$$

One of the constants may be chosen arbitrarily.

The ratio $a/b$ has two values corresponding to the two roots of $\beta$:

$$\frac{a_1}{b_1} = 1 - \beta_2 \quad \frac{a_2}{b_2} = 1 - \beta_1$$

Substituting the values found for $\beta$ we get:

$$\frac{a_1}{b_1} = 1 - \varepsilon \frac{h'}{h + h'} \quad \frac{a_2}{b_2} = -\frac{h'}{h} + \varepsilon \frac{h'}{h + h'}$$
Finally, the two sets of independent transport equations corresponding to the solutions for $\beta$ are:

\[
\begin{align*}
\frac{\partial U_k}{\partial t} - f V_k &= -g \beta_k h \frac{\partial \eta_k}{\partial x} + \left( \frac{\tau_w^{(x)}}{\rho} \right)_k \\
\frac{\partial V_k}{\partial t} + f U_k &= -g \beta_k h \frac{\partial \eta_k}{\partial y} + \left( \frac{\tau_w^{(y)}}{\rho} \right)_k \\
\frac{\partial U_k}{\partial x} + \frac{\partial V_k}{\partial y} &= -\frac{\partial \eta_k}{\partial t}
\end{align*}
\]

Where:

\[
\begin{align*}
U_k &= a_k U + b_k U' \\
V_k &= a_k V + b_k V' \\
\eta_k &= a_k \eta + (b_k - a_k) \eta' \\
\left( \frac{\tau_w^{(x)}}{\rho} \right)_k &= a_k \frac{\tau_w^{(x)}}{\rho} \\
\left( \frac{\tau_w^{(y)}}{\rho} \right)_k &= a_k \frac{\tau_w^{(y)}}{\rho}
\end{align*}
\]

With: $k = (1, 2)$