**Exercise 1.** Solve the following Cauchy problem, by using the well-known Laplace-transform method (you can benefit from the formulas in the back of this paper):

\[
\begin{aligned}
&\begin{cases}
  u'' + 2u' - 3u = 6te^{-t} \\
  u(0) = 0 \\
  u'(0) = 1
\end{cases}
\end{aligned}
\]

**Solution:** By applying the Laplace transform respectively to the left-hand side and to the right-hand side of the ODE, we get

\[
(s^2 + 2s - 3) \cdot U(s) = 6 \quad \text{and} \quad \frac{6}{(s+1)^2},
\]

where \(U(s)\) denote \(L[u](s)\). By equating both sides, and solving for \(U\), we get

\[
U(s) = \frac{s^2 + 2s + 7}{(s + 3)(s - 1)(s + 1)^2} = A \cdot \frac{s}{s + 3} + B \cdot \frac{1}{s - 1} + C \cdot \frac{1}{s + 1} + D \cdot \frac{1}{(s + 1)^2},
\]

where \(A, \ldots, D\) are constants to be determined. Since \(A\) and \(B\) are residues at simple poles, since \(C\) is a residue at a pole of order 2, and since \(D\) is a coefficient which we called “of type \(a_{-2}\)” (relative to the pole \(-1\)) during our lessons, from well-known formulas we get:

\[
\begin{aligned}
A &= \text{Res}(U, -3) = \frac{s^2 + 2s + 7}{(s - 1)(s + 1)^2} \bigg|_{s = -3} = [...] = -\frac{5}{8}; \\
B &= \text{Res}(U, 1) = \frac{s^2 + 2s + 7}{(s + 3)(s + 1)^2} \bigg|_{s = 1} = [...] = \frac{5}{8}; \\
C &= \text{Res}(U, -1) = \frac{d}{ds} \bigg|_{s = -1} \left( \frac{s^2 + 2s + 7}{(s + 3)(s - 1)} \right) = -\frac{20(s + 1)}{(s^2 + 2s - 3)^2} = 0; \\
D &= \frac{s^2 + 2s + 7}{(s + 3)(s - 1)} \bigg|_{s = -1} = [...] = -\frac{3}{2}.
\end{aligned}
\]

Thus

\[
U(s) = \frac{-\frac{5}{2}}{s + 3} + \frac{\frac{5}{8}}{s - 1} + \frac{0}{s + 1} + \frac{-\frac{3}{2}}{(s + 1)^2}.
\]

By means of Laplace anti-transform \(L^{-1}\), we finally get

\[
u(t) = \frac{5}{8} e^{-3t} + \frac{5}{8} e^t - \frac{3}{2} t e^{-t}.
\]
**Exercise 2.** Find the Fourier transform of the following function:

\[ f(x) = \frac{1}{(x + i)(x - 3i)^2} \]

carefully explaining your arguments.

**Solution:** By definition, one has

\[ \hat{f}(\xi) = \int_{-\infty}^{+\infty} g(x) \, dx, \quad \text{where we set } g(x) := \frac{e^{-i\xi x}}{(x + i)(x - 3i)^2}. \]

We observe that \( z = -i \) is a pole of order 1 for \( g(z) \), and \( z = 3i \) is a pole of order 2 for \( g(z) \). We also remark that \( f \) is continuous on the real axis, and \( |f(z)| \approx \frac{1}{|z|^2} \) as \( |z| \to \infty \), so that Jordan’s Lemma (in the version that we saw during the lessons) can be applied. We exploit the well-know geometric method based on the use of the curve obtained by joining together the real segment \([-R, R]\) and

**(when \( \xi > 0 \)):** the semi-circle of centre the origin and radius \( R \) in the half-plane \( \{z : \Im(z) \leq 0\} \) (clock-wise oriented),

**(when \( \xi < 0 \)):** the semi-circle of centre the origin and radius \( R \) in the half-plane \( \{z : \Im(z) \geq 0\} \) (counter-clock-wise oriented).

Upon the Residue Theorem, and an inversion of the orientation of the curve if the case \( \xi > 0 \) (in order to obtain a positively-oriented path bounding a regular domain), we get

**(when \( \xi > 0 \)):**

\[ \hat{f}(\xi) = -2\pi i \operatorname{Res}(g(z), -i) = -2\pi i \left. \frac{e^{-i\xi z}}{(z - 3i)^2} \right|_{z=-i} = [... \right] = \frac{\pi i}{8} e^{-\xi}; \]

**(when \( \xi < 0 \)):**

\[ \hat{f}(\xi) = 2\pi i \operatorname{Res}(g(z), 3i) = \left. 2\pi i \frac{d}{dz} \left( \frac{e^{-i\xi z}}{z + i} \right) \right|_{z=3i} = [... \right] = \frac{\pi i}{8} e^{3\xi} (1 - 4\xi). \]

The case \( \xi = 0 \) can be obtained as a limit case for \( \xi \to 0^{\pm} \) (since \( \hat{f} \) is continuous) so that

\[ \hat{f}(\xi) = \begin{cases} 
\frac{\pi i}{8} e^{-\xi} & \text{when } \xi > 0 \\
\frac{\pi i}{8} & \text{when } \xi = 0 \\
\frac{\pi i}{8} e^{3\xi} (1 - 4\xi) & \text{when } \xi < 0.
\end{cases} \]
Exercise 3. Solve the following problems (and mention the formulae you are applying):

(3.i) Count the anagrams (possibly meaningless) of the word LAPLACE

(3.ii) Mary has 6 pupils, and she wants to award them with 5 gold stars. In how many ways can she award them, in the possibility that some pupils obtain no star at all?

(3.iii) We have two dice: one of them is loaded and, at every launch, it gives a 5; the other is unloaded. Rolling one of them at random, we get a 5. What is the probability that we have launched the loaded dice?

Solution:

(3.i) We use permutations of 7 elements (namely the letters L, A, P, L, A, C, E) of which 2 are equal to each other (the A’s), and another 2 are equal to each other (the L’s); by the known formula, the needed number is

\[ P^{(2,2)}_7 \frac{7!}{2!2!} = [\ldots] = 1260. \]

(3.ii) This is an exercise using combinations with repetitions of the 5 stars for the 6 pupils; the needed number is

\[ C'_{5,6} = \binom{5+6-1}{6-1} = \binom{10}{5} = [\ldots] = 252. \]

(3.iii) We use Bayes’ Theorem; if we define the events

\textbf{event } E: \text{ “we got a 5”}

\textbf{event } E_1: \text{ “the loaded dice has been rolled”}

\textbf{event } E_2: \text{ “the unloaded dice has been rolled”,}

we need \( p(E_1|E) \), which is given by

\[ p(E_1|E) = \frac{p(E|E_1) \cdot p(E_1)}{p(E|E_1) \cdot p(E_1) + p(E|E_2) \cdot p(E_2)} = \frac{1 \cdot \frac{1}{2}}{\frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2}} = [\ldots] = \frac{6}{7}. \]

The same result can also be obtained by counting favourable cases over possible cases in the probability sub-space \( E \).