DIGITAL CONTROL SYSTEMS INTRODUCTION
Typical digital control systems feedback loop

Sampling of the error signal:

Sampling of the measure:
Signal classification

(a) Analogical signal; b) Quantized signal; c) Sampled signal; d) Digital signal
A/D, Analog/Digital converter:

\[ x(t) \xrightarrow{A/D} x(kT) \]

Model: impulsive sampling

\[ x(t) \xrightarrow{\delta(t - kT)} x(kT)\delta(t - kT) \]
Interface devices

D/A, Digital/Analog converter

\[ x(kT) \rightarrow D/A \rightarrow x_r(t) \]

Model: zero order hold

\[ x(kT) \rightarrow x(kT)\delta(t - kT) \rightarrow G_r(s) \rightarrow x_r(t) \]
MATHEMATICAL TOOLS
Difference equations

\[ u_k = f(e_0, e_1, \ldots, e_k; u_0, u_1, \ldots, u_{k-1}) \]

If \( f(\cdot) \) is linear:

\[ u_k = -a_1 u_{k-1} - \ldots - a_n u_{k-n} + b_0 e_k + \ldots + b_m e_{k-m} \]

Example:

\[ u_k = -a_1 u_{k-1} - a_2 u_{k-2} + b_0 e_k \]

Defining \( \nabla \) as the delay operator

\[
\begin{align*}
  u_k &= u_k \\
  u_{k-1} &= u_k - \nabla u_k \\
  u_{k-2} &= u_k - 2\nabla u_k + \nabla^2 u_k
\end{align*}
\]

we obtain

\[ a_2 \nabla^2 u_k - (a_1 + 2a_2) \nabla u_k + (a_2 + a_1 + 1) u_k = b_0 e_k \]
Solution of a difference equation with constant coefficients

\[ u_k = u_{k-1} + u_{k-2} \quad k \geq 2 \]

\[ u_0 = u_1 = 1. \]
Difference equations

The elementary solution is in the form of $z^k$:

$$cz^k = cz^{k-1} + cz^{k-2}$$

$$z^2 - z - 1 = 0$$

$$z_{1,2} = \left(1 \pm \sqrt{5}\right)/2$$

In general it holds:

$$u_k = c_1 z_1^k + c_2 z_2^k$$

with $c_1, c_2$ to be computed using initial conditions for $k = 0, 1$. In previous case:

$$u_k = \frac{1 + \sqrt{5}}{2\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^k + \frac{-1 + \sqrt{5}}{2\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^k$$

The trend is diverging, hence the system is unstable.

If all the roots of the characteristic equations are within the unitary circle, then the corresponding difference equation is stable, i.e., its solution will converge to zero in time for any finite initial condition.
The Z-Transform

Given a sequence $x_k \in \mathbb{R}$, defined for $k = 0, 1, 2, \ldots$ and null for $k < 0$. The $Z$-transform of $x_k$ is a function of the complex variable $z$ defined as

$$X(z) = Z[x_k] = x_0 + x_1 z^{-1} + \cdots + x_k z^{-k} + \cdots = \sum_{k=0}^{\infty} x_k z^{-k}$$

In the case of a sequence $x_k$ obtained by uniformly sampling a continuous signal $x(t)$, $t \geq 0$ with a sampling time $T$, then $x_k = x(kT)$:

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

The extended equation

$$X(z) = x(0) + x(T) z^{-1} + x(2T) z^{-2} + \cdots + x(kT) z^{-k} + \cdots$$

implies the specification of the **sampling time** $T$, from which the samples depends (i.e., the coefficients of the series).
**The Z-transform**

We write: $X(z) = \mathcal{Z}[X(s)]$ meaning $X(z) = \mathcal{Z}\left[\mathcal{L}^{-1}[X(s)]\big|_{t=kT}\right]$

In engineering applications, the function $X(z)$ assumes in general a rational fractional expression

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$

that can also be expressed in powers of $z^{-1}$:

$$X(z) = \frac{z^n \left(b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}\right)}{z^n \left(1 + a_1 z^{-1} + \cdots + a_n z^{-n}\right)}$$

$$= \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Example:

$$X(z) = \frac{z(z + 0.5)}{(z + 1)(z + 2)} = \frac{1 + 0.5 z^{-1}}{(1 + z^{-1})(1 + 2 z^{-1})}$$
Z-Transform of elementary terms

Unitary discrete impulse. Kronecker’s function $\delta_0(t)$:

$$x(t) = \begin{cases} 
1 & t = 0 \\
0 & t \neq 0
\end{cases}$$

$$X(z) = \mathcal{Z}[x(t)] = \sum_{k=0}^{\infty} x(kT)z^{-k} = 1 + 0z^{-1} + 0z^{-2} + 0z^{-3} + \cdots = 1$$

Unitary step:

$$x(t) = h(t) = \begin{cases} 
1 & t \geq 0 \\
0 & t < 0
\end{cases} \quad \text{i.e.} \quad h(k) = \begin{cases} 
1 & k = 0, 1, 2, \ldots \\
0 & k < 0
\end{cases}$$

$$H(z) = \mathcal{Z}[h(t)] = \sum_{k=0}^{\infty} h(kT)z^{-k} = \sum_{k=0}^{\infty} z^{-k} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \quad \text{. The series converges for } |z| > 1 \quad .$$
Unitary ramp:

\[
x(t) = \begin{cases} 
  t & t \geq 0 \\
  0 & t < 0
\end{cases}
\]

Since \( x(kT) = kT \), \( k = 0, 1, 2, \ldots \), the \( Z \)-transform is

\[
X(z) = \mathcal{Z}[x](z) = \sum_{k=0}^{\infty} x(kT)z^{-k} = T \sum_{k=0}^{\infty} k z^{-k}
\]

\[
= T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots)
\]

\[
= Tz^{-1}(1 + 2z^{-1} + 3z^{-2} + \cdots)
\]

\[
= T \frac{z^{-1}}{(1 - z^{-1})^2} = T \frac{z}{(z - 1)^2}
\]

converging for \(|z| > 1\).
Z-Transform of elementary terms

Exponential function:

\[
x(t) = \begin{cases} 
e^{-at} & t \geq 0 \\
0 & t < 0
\end{cases}
\]

where \(a\) is a real or complex constant. Since \(x(kT) = e^{-akT}, k = 0, 1, 2, \ldots\), we have

\[
X(z) = \mathcal{Z}[e^{-at}] = \sum_{k=0}^{\infty} e^{-akT} z^{-k}
\]

\[
= 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \ldots
\]

\[
= \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}
\]

that converges for \(|z| > e^{-\text{Re}(a)T}\). Note that for \(a = 0\) we move back to the unitary step.
Z-Transform of elementary terms

Sinusoidal function:

\[ x(t) = \begin{cases} 
\sin \omega t & t \geq 0 \\
0 & t < 0 
\end{cases} \]

From Euler’s equations

\[ \sin \omega t = \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) \]

Thus,

\[ X(z) = Z[\sin \omega t] = \frac{1}{2j} \left( \frac{1}{1 - e^{j\omega T}z^{-1}} - \frac{1}{1 - e^{-j\omega T}z^{-1}} \right) \]

\[ = \frac{1}{2j} \frac{(e^{j\omega T} - e^{-j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \]

\[ = \frac{z^{-1} \sin \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}} = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \]

converging for \(|z| > 1\).
Z-Transform of elementary terms

Cosinusoidal function:

\[
x(t) = \begin{cases} 
\cos \omega t & t \geq 0 \\
0 & t < 0 
\end{cases}
\]

\[
X(z) = Z[\cos \omega t] = \frac{1}{2} \left( \frac{1}{1 - e^{j\omega T} z^{-1}} + \frac{1}{1 - e^{-j\omega T} z^{-1}} \right)
\]

\[
= \frac{1}{2} \frac{2 - (e^{-j\omega T} + e^{j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}}
\]

\[
= \frac{1 - z^{-1} \cos \omega T}{1 - 2z^{-1} \cos \omega T + z^{-2}}
\]

\[
= \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1} \quad |z| > 1
\]
Z-Transform of elementary terms

Example:

\[ X(s) = \frac{1}{s(s + 1)} \]

First technique:

\[ x(t) = 1 - e^{-t} \]

\[ X(z) = \mathcal{Z}[1 - e^{-t}] = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}} = \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})} = \frac{(1 - e^{-T})z}{(z - 1)(z - e^{-T})} \]

Second technique:

\[ X(s) = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{1 + s} \]

\[ X(z) = \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}} \]
The Z-transform $X(z)$ and the corresponding sequence $x(k)$ are in one-to-one correspondance.

This is not true for the Z-transform $X(z)$ and its inverse $x(t)$.

From a $X(z)$ it is possible to obtain many $x(t)$.

This ambiguity does not hold if restrictive conditions on the sampling time $T$ hold (Shannon’s theorem).

Different continuous time functions can have the same samples: $x(k)$.
Properties of the Z-transform

♠ Linearity:

\[ x(k) = af(k) + bg(k) \]
\[ X(z) = aF(z) + bG(z) \]

♠ Multiplication for \( a^k \):

Being \( X(z) \) the Z-transform of \( x(t) \), \( a \) a constant value.

\[ Z[a^k x(k)] = X(a^{-1}z) \]

\[
Z[a^k x(k)] = \sum_{k=0}^{\infty} a^k x(k) z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} \\
= X(a^{-1}z)
\]
Properties of the Z-transform

♠ Time shifting:

If \( x(t) = 0, t < 0 \), \( X(z) = \mathcal{Z}[x(t)] \), and \( n = 1, 2, \ldots \), then

\[
\mathcal{Z}[x(t - nT)] = z^{-n} X(z) \quad \text{(delay)}
\]

\[
\mathcal{Z}[x(t + nT)] = z^n \left[ X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \quad \text{(anticipation)}
\]

Note that:

\[
z^{-1} x(k) = x(k - 1)
\]

\[
z^{-2} x(k) = x(k - 2)
\]

\[
z x(k) = x(k + 1)
\]
Properties of the Z-transform

♠ Delay:

\[ \mathcal{Z}[x(t - nT)] = \sum_{k=0}^{\infty} x(kT - nT)z^{-k} \]

\[ = z^{-n} \sum_{k=0}^{\infty} x(kT - nT)z^{-(k-n)} \]

defining \( m = k - n \),

\[ \mathcal{Z}[x(t - nT)] = z^{-n} \sum_{m=-n}^{\infty} x(mT)z^{-m} \]

Since \( x(mT) = 0 \) for \( m < 0 \), we can write

\[ \mathcal{Z}[x(t - nT)] = z^{-n} \sum_{m=0}^{\infty} x(mT)z^{-m} = z^{-n} X(z) \]
Properties of the Z-transform

♠ Anticipation:

\[ \mathcal{Z}[x(t + nT)] = \sum_{k=0}^{\infty} x(kT + nT)z^{-k} = z^n \sum_{k=0}^{\infty} x(kT + nT)z^{-(k+n)} \]

\[ = z^n \left[ \sum_{k=0}^{\infty} x(kT + nT)z^{-(k+n)} + \sum_{k=0}^{n-1} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \]

\[ = z^n \left[ \sum_{k=0}^{\infty} x(kT)z^{-k} - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \]

\[ = z^n \left[ X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k} \right] \]
♠ Initial value theorem: If $X(z)$ is the $Z$-transform of $x(t)$ and if

$$
\lim_{z \to \infty} X(z)
$$

exists, then the initial value $x(0)$ of $x(t)$ is given by:

$$
x(0) = \lim_{z \to \infty} X(z)
$$

In fact, note that

$$
X(z) = \sum_{k=0}^{\infty} x(k)z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots
$$
Final value theorem: being all the poles of $X(z)$ within the unitary circle with at most a simple pole in $z = 1$.

$$\lim_{k \to \infty} x(k) = \lim_{z \to 1} [(1 - z^{-1})X(z)]$$

In fact:

$$\sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} = X(z) - z^{-1}X(z)$$

$$\lim_{z \to 1} \left[ \sum_{k=0}^{\infty} x(k)z^{-k} - \sum_{k=0}^{\infty} x(k-1)z^{-k} \right] =$$

$$= \sum_{k=0}^{\infty} [x(k) - x(k-1)]$$

$$= [x(0) - x(-1)] + [x(1) - x(0)] + [x(2) - x(1)] + \cdots$$

$$= \lim_{k \to \infty} x(k)$$
*♠* Real convolution theorem:
given two functions \( x_1(t) \) e \( x_2(t) \), with \( x_1(t) = x_2(t) = 0, \ t < 0 \) and \( Z \)-transform \( X_1(z) \), \( X_2(z) \). Then

\[
X_1(z)X_2(z) = Z \left[ \sum_{h=0}^{k} x_1(hT)x_2(kT - hT) \right]
\]

Note that

\[
Z \left[ \sum_{h=0}^{k} x_1(h)x_2(k - h) \right] = \sum_{k=0}^{\infty} \sum_{h=0}^{k} x_1(h)x_2(k - h)z^{-k} = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} x_1(h)x_2(k - h)z^{-k}
\]

Since \( x_2(k - h) = 0, \ h > k \). Defining \( m = k - h \) we have

\[
Z \left[ \sum_{h=0}^{k} x_1(h)x_2(k - h) \right] = \sum_{h=0}^{\infty} x_1(h)z^{-h} \sum_{m=0}^{\infty} x_2(m)z^{-m}
\]
The inverse Z-transform

Let to obtain a sequence $x_k$ (and possibly the continuous function $x(t)$ whose samples are $x_k$) from the $Z$-transform $X(z)$.

\[ X(z) \quad \text{one-to-one} \quad \rightarrow \quad x(k) \quad \text{non one-to-one} \quad \rightarrow \quad x(t) \]

If the \textbf{Shannon’s theorem} on sampling holds, then the continuous time function $x(t)$ can be univocally derived from the sequence $x_k$. 
The inverse Z-transform: decomposition in simple fractions

\[ X(z) = \frac{b_0z^m + b_1z^{m-1} + \cdots + b_{m-1}z + b_m}{(z - p_1)(z - p_2) \cdots (z - p_n)} \]

Case 1: All the poles are simple

\[ X(z) = \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \cdots + \frac{c_n}{z - p_n} = \sum_{i=1}^{n} \frac{c_i}{z - p_i} \]

residue \( c_i \) are computed as: \( c_i = \left[ (z - p_i)X(z) \right]_{z=p_i} \).

If \( X(z) \) has a zero in the origine, the function \( \frac{X(z)}{z} \) must be used

\[ \frac{X(z)}{z} = \frac{c_1}{z - p_1} + \cdots + \frac{c_n}{z - p_n} \quad c_i = \left[ \frac{(z - p_i)X(z)}{z} \right]_{z=p_i} \]

When we have complex conjugated poles, also the coefficients \( c_i \) are complex number. In this case use Euler's equations to obtain trigonometry functions.

The mathematical expression of the inverse transform is

\[ x(k) = \sum_{i=1}^{n} c_i p_i^k \]
Case 2: If \( X(z) \), or \( X(z)/z \), has multiple poles

\[
X(z) = \frac{B(z)}{A(z)} = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z - p_1)^{r_1} (z - p_2)^{r_2} \cdots (z - p_h)^{r_h}}
\]

then

\[
X(z) = \sum_{i=1}^{h} \sum_{k=1}^{r_i} \frac{c_{ik}}{(z - p_i)^{r_i-k+1}}
\]

where residues can be computed as

\[
c_{ik} = \left[ \frac{1}{(k - 1)!} \frac{d^{k-1}}{dz^{k-1}} (z - p_i)^{r_i} X(z) \right]_{z=p_i}
\]

\[i = 1, \ldots, h; \quad k = 1, \ldots, r_i\]
The inverse Z-transform: decomposition in simple fractions

Example:

\[ X(z) = \frac{1}{z^4 + 6z^3 + 13z^2 + 12z + 4} = \frac{1}{(z + 2)^2(z + 1)^2} \]

We have:

\[ X(z) = \frac{c_{11}}{(z + 2)^2} + \frac{c_{12}}{(z + 2)} + \frac{c_{21}}{(z + 1)^2} + \frac{c_{22}}{(z + 1)} \]

\[ c_{11} = \left[ (z + 2)^2 X(z) \right]_{z=-2} = 1 \]

\[ c_{12} = \left[ \frac{d}{dz} (z + 2)^2 X(z) \right]_{z=-2} = 2 \]

\[ c_{21} = \left[ (z + 1)^2 X(z) \right]_{z=-1} = 1 \]

\[ c_{22} = \left[ \frac{d}{dz} (z + 1)^2 X(z) \right]_{z=-1} = -2 \]
SAMPLING AND INVERSE SAMPLING
Digital feedback systems are characterized by a **continuous time part** (the plant) and a **discrete time part** (the digital controller)

Hence both **continuous time variables** and **discrete time variables** coexist

Interface devices are the **sampler** and the **inverse sampler**

Zero order hold (hold inverse sampling):

\[
x_r(t) = \sum_{k=0}^{\infty} x(kT)[h(t - kT) - h(t - (k + 1)T)]
\]

\[
X_r(s) = \sum_{k=0}^{\infty} x(kT) \left[ \frac{e^{-kTs} - e^{-(k+1)Ts}}{s} \right] = \frac{1 - e^{-Ts}}{s} \sum_{k=0}^{\infty} x(kT)e^{-kTs}
\]
Impulsive sampling

\[ H_0(s) = \frac{1 - e^{-Ts}}{s} \quad \text{and} \quad X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs} \]

\[ x^*(t) = \mathcal{L}^{-1}[X^*(s)] = \sum_{k=0}^{\infty} x(kT)\delta(t - kT) \]

\[ \delta_T(t) = \sum_{k=0}^{\infty} \delta(t - kT) \]
**Impulsive sampling**

The **impulsive sampler** is an ideal model of the real sampler (A/D converter) used to analyze and design digital control systems.

The output of the zero order hold is:

\[ X_r(s) = H_0(s) X^*(s) = \frac{1 - e^{-Ts}}{s} X^*(s) \]

\[ x(t) \quad \text{Hold} \quad x_r(t) \quad \delta_T \]

\[ x(t) \quad x^*(t) \quad 1 - e^{-Ts} \quad 1 \]

\[ X^*(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs} \]

\[ z = e^{sT} \quad \iff \quad s = \frac{1}{T} \ln z \quad \Rightarrow \quad X^*(s) \bigg|_{s = \frac{1}{T} \ln z} = \sum_{k=0}^{\infty} x(kT) z^{-k} \]
Compute the Laplace transform of the sampled signal $x^*(t)$:

$$x^*(t) = x(t) \delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \quad \text{con} \quad c_n = \frac{1}{T} \int_{0}^{T} \delta_T(t) e^{-jn\omega_s t} dt = \frac{1}{T}$$

which is

$$x^*(t) = x(t) \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(t) e^{jn\omega_s t}$$

$$X^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \mathcal{L} [x(t) e^{jn\omega_s t}] = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(s - jn\omega_s)$$

Disregarding the gain $1/T$, the Laplace transform of the sampled signal $X^*(s)$ is the sum of infinite terms, $X(s - jn\omega_s)$, each of them obtained by a $jn\omega_s$ shifting of $X(s)$.
Impulsive sampling

Hence the spectrum of the sampled signal is:

\[ X^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\omega - j n\omega_s) \]
The condition $\omega_s > 2\omega_c$ lets the spectrum of the main spectral component be divided by the repetitions. Hence, through a filtering operation, it is possible to exactly recover the original signal $x(t)$ from the sampled signal $x^*(t)$.

In case the condition $\omega_s > 2\omega_c$ does not hold:

The main spectral component is partially superimposed to its repetitions, hence it is not possible to isolate it recovering the original signal.
Shannon’s theorem

Let $\omega_s = \frac{2\pi}{T}$ be the sampling pulse ($T$ is the sampling time), and let $\omega_c$ be the higher spectral component of the continuous time signal $x(t)$. Signal $x(t)$ can be recovered starting from the sampled signal $x^*(t)$ if and only if:

$$\omega_s > 2\omega_c$$

The perfect recovery can be pursued using the ideal filter:

$$G_I(j\omega) = \begin{cases} 
T & -\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2} \\
0 & \text{elsewhere}
\end{cases}$$
The ideal filter $G_I(j\omega)$ is not feasible; in fact, its impulsive response is:

$$g_I(t) = \frac{\sin(\omega_s t/2)}{\omega_s t/2}$$

This means that the recovered signal is

$$x(t) = \int_{-\infty}^{\infty} x^*(\tau) g_I(t - \tau) d\tau = \sum_{k=-\infty}^{\infty} x(kT) \int_{-\infty}^{\infty} \delta(\tau - kT) \frac{\sin(\omega_s (t - \tau)/2)}{\omega_s (t - \tau)/2} d\tau$$

$$= \sum_{k=-\infty}^{\infty} x(kT) \frac{\sin(\omega_s (t - kT)/2)}{\omega_s (t - kT)/2}$$

We need all the past and future samples $x(kT)$!!

We will use approximated feasible inverse sampler.
The aliasing phenomenon

With the term **aliasing** we intend the generation of new spectral components, due to the sampling operation, partially superimposed to the main component. These new components do not allow the exact recovery of the original signal. The aliasing appears just if the Shannon’s theorem condition $\omega_s > 2\omega_c$ is not met.

Example: consider

\[
\begin{align*}
    x(t) & = \sin(\omega_2 t + \theta) \\
    y(t) & = \sin((\omega_2 + n\omega_s)t + \theta)
\end{align*}
\]

having the same phase $\theta$ and pulses that differ for a integer multiple of $\omega_s$.

If the signals are sampled

\[
\begin{align*}
    x(kT) & = \sin(\omega_2 kT + \theta) \\
    y(kT) & = \sin((\omega_2 + n\omega_s)kT + \theta) \\
    & = \sin(\omega_2 kT + 2k\pi n + \theta) \\
    & = \sin(\omega_2 kT + \theta)
\end{align*}
\]

samples are the same: $x(kT) = y(kT)$
Example: $\omega_2 + \omega_1 = n\omega_s$

Being $\omega_1 = \frac{1}{8}\omega_s$ e $\omega_2 = \omega_s - \omega_1 = \frac{7}{8}\omega_s$

\[
\begin{align*}
  x(t) &= \sin(\omega_1 t) = \sin(\omega_s t/8) \\
  y(t) &= \sin(\omega_2 t) = \sin(7\omega_s t/8 + \pi)
\end{align*}
\]

Sampling we have

\[
\begin{align*}
  x(kT) &= \sin(\omega_s k T/8) = \sin(2k\pi/8) = \sin(k\pi/4) \\
  y(kT) &= \sin(7\omega_s kT/8 + \pi) = \sin(7k\pi/4 + \pi) = \sin(k\pi/4)
\end{align*}
\]

To avoid aliasing it is important to opportunealy filter the signal \textbf{before} the sample: \textbf{anti aliasing filters}.
Sampling example

Sampling of the impulsive response of the second order system

\[ G(s) = \frac{25}{s^2 + 6s + 25} \]

Unitary DC gain, complex conjugated poles \( p_{1,2} = -3 \pm j4 \), natural pulse \( \omega_n = 5 \text{ rad/s} \) and damping coefficient \( \delta = 3/5 \).

Amplitude Bode diagram of \( G(j\omega) \):

For \( \omega > 10\omega_n = 50 \text{ rad/s} = \bar{\omega} \), the module of \( G(j\omega) \) is below 1/100 (-40 db) of the DC gain.
Sampling example

The spectrum, even if ideally has spectral components until unbounded frequency, can be neglected for \( \omega \geq \bar{\omega} = 50 \text{ rad/s} \)

Applying the \( \mathcal{Z} \)-transform we have

\[
G(z) = \frac{25}{4} \frac{e^{-3T} \sin(4T)}{z^2 - 2e^{-3T} \cos(4T)} z + e^{-6T}
\]

We can compute the spectral behavior as \( G^*(j\omega) = G(z)|_{z = e^{j\omega T}} \) \( (0 \leq \omega \leq \frac{\pi}{T}) \)

Different trends for \( T = \frac{\pi}{50} \) and \( T = \frac{\pi}{25} \):

![Graph](image-url)
**Typical inverse samplers:**

\[
x(t) = x(kT) + \frac{dx(t)}{dt} \bigg|_{t=kT} (t - kT) + \frac{d^2 x(t)}{dt^2} \bigg|_{t=kT} \frac{(t - kT)^2}{2!} + \cdots
\]

\[
\left. \frac{dx(t)}{dt} \right|_{t=kT} \approx \frac{x(kT) - x((k-1)T)}{T}
\]

**Zero order hold:**

\[x_0(t) = x(kT)\]  \quad \text{for} \quad kT \leq t < (k+1)T

\[g_0(t)\]

\[H_0(s) = \frac{1 - e^{-sT}}{s}\]
Zero order hold

The frequential response of the zero order hold is:

\[ H_0(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega} = \frac{2e^{-j\omega T/2}}{\omega} \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \]

\[ = T \frac{\sin(\omega T/2)}{\omega T/2} e^{-j\omega T/2} \]

Module

\[ |H_0(j\omega)| = T \left| \frac{\sin(\omega T/2)}{\omega T/2} \right| \approx T \quad \text{for } \omega \ll \omega_s = 2\pi/T \]

Phase

\[ Arg[H_0(j\omega)] = Arg \left[ \sin \frac{\omega T}{2} \right] - \frac{\omega T}{2} \approx -\frac{\omega T}{2} \quad \text{for } \omega \ll \omega_s = 2\pi/T \]
Zero order hold

Scala lineare

Scala logaritmica

Fase (deg)

W/\omega_s

W/\omega_s

|H(j\omega)/T| (db)

|H(j\omega)/T|
Zero order hold

\[ H_0(j\omega) \approx T e^{-j\omega T/2} \]
Antialiasing filters

The aliasing produced by the sampling introduces undesired harmonic components within the bandwidth of the closed loop system.

We need to introduce filters that attenuate the more as possible the noise.
Antialiasing filters

(a)

(b)

(c)

(d)
Antialiasing filters

Analogical filters: passive or actives
- RC filters (first order)
- Slope 20 $db$ per decade
- The filter should not perturb the frequencies at which system works

Digital filters
- Sampling time lower then the sampling time of the controller
- Mean filter:

$$y(k) = \frac{1}{N} \sum_{i=0}^{N-1} u(k - i)$$

with sampling time

$$T_d = \frac{T_c}{N}$$
Antialiasing filters

Bode diagrams of Bode filters
Antialiasing filters

How to chose time constants for the analog and digital filters

$$\omega_a = \frac{\omega_c}{2} = \frac{\pi}{T_c}$$, \quad $$\omega_a \approx \frac{2}{T_c}$$, \quad $$\tau_a = \frac{1}{\omega_a} \approx \frac{T_c}{2}$$

$$\tau_d \approx \frac{T_c}{2}$$, while the time constant of the analog filter $$\tau_a$$ must be computed according to the sampling time of the filter: $$\tau_a \approx \frac{T_d}{2}$$
Antialiasing filters: example

\[ G(s) = \frac{2}{s(s+1)(s+2)} \quad D(z) = \frac{0.94527(z - 0.97884)(z - 0.92433)}{(z - 0.80687)(z - 0.99216)} \]

Response without antialiasing filtering. Gaussian noise with variance 0.1
Antialiasing filters: example

Antialiasing first order filter with $1/\tau = 5 \text{ rad/s}$ ($1/\tau = 2 \text{ rad/s}$)

Drawback: system slows down and overshoot increase
Antialiasing filters: example

Sinusoidal noise with amplitude 0.5 and frequency $f = 0.3183 \ Hz = 2 \ rad/s$

Selective filter (notch): $\omega = 2 \ rad/s$

$$F(s) = \frac{s^2 + 4}{s^2 + 0.4s + 4}$$
Since

\[ X^*(s) = X(z)|_{z = e^{sT}} \]

variables \( s \) and \( z \) are linked by the relation \( z = e^{sT} \)

Being \( s = \sigma + j\omega \) we have

\[ z = e^{T(\sigma+j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} e^{jT(\omega + \frac{2k\pi}{T})} \]

**Important**: any point in \( z \) is in correspondence with infinite points if the plane \( s \).

The points in \( s \) with negative real part (\( \sigma < 0 \)) are in correspondence with the points in \( z \) within the unitary circle

\[ |z| = e^{T\sigma} < 1 \]

The points in \( s \) on the imaginary axis (\( \sigma = 0 \)) are mapped on the unitary circle (\( |z| = 1 \)), while those with positive real part (\( \sigma > 0 \)) are mapped outside the unitary circle (\( |z| > 1 \)).

The strip of the \( s \) plane bounded by the horizontal lines \( s = j\omega_s/2 \) and \( s = -j\omega_s/2 \) is called **main strip**.
Plane $s$ and plane $z$ correspondence

Main strip and complementary strips

$s$ plane

$z$ plane

Automatic control 2 - Digital control systems – p. 57/139
Complex variables $s$ and $z$ are linked by the relation $z = e^{sT}$.

Being $s = \sigma + j\omega$ we have

$$z = e^{T(\sigma+j\omega)} = e^{T\sigma} e^{jT\omega} \quad \left(0 \leq \omega \leq \frac{\omega_s}{2} = \frac{\pi}{T}\right)$$

Mapping between main strip and $z$ plane
Plane $s$ and plane $z$ correspondence

Constant exponential decay loci ($\sigma =$const)

\[
e^{\sigma_1 T} \quad e^{-\sigma_1 T} \quad e^{\sigma_2 T}
\]
Constant pulse loci ($\omega =$ const)
**Plane $s$ and plane $z$ correspondence**

Example:

![Diagram showing the correspondence between the $s$-plane and the $z$-plane. The diagram includes the real and imaginary axes for both planes, with key points marked and equations for $z$ plane calculations.](image)

\[
\begin{align*}
\sigma &= -\sigma_2 \\
\sigma &= -\sigma_1 \\
\omega_2 &= j\omega_2 \\
\omega_1 &= j\omega_1 \\
\sigma &= -\sigma_1 \\
\omega_2 &= j\omega_2 \\
\omega_1 &= j\omega_1 \\
Z &= e^{T(\sigma + j\omega_1)} \\
Z &= e^{T(\sigma - j\omega_2)} \\
\end{align*}
\]
Constant damping coefficient $\delta$ and natural pulse $\omega_n$ loci in $z$ plane
The points in $s$ and in $z$, such that $z = e^{sT}$, can be considered as the poles of the corresponding Laplace transform $F(s)$ and $Z$-transform $F(z)$, with $F(z)$ obtained by sampling $F(s)$.
Transient responses in $z$ plane
**Discrete Time Systems**
Continuous time systems

\[ x(t) \rightarrow \begin{array}{c} g(t) \\ G(s) \end{array} \rightarrow c(t) = \int_{0}^{t} g(\tau)x(t - \tau)\,d\tau \]

\[ X(s) \rightarrow \begin{array}{c} G(s) \end{array} \rightarrow C(s) = G(s)\,X(s) \]

Discrete time systems

\[ e_{k} \rightarrow \begin{array}{c} d_{k} \\ D(z) \end{array} \rightarrow m_{k} = \sum_{i=0}^{k} d_{i}e_{k-i} = \sum_{i=0}^{k} e_{i}d_{k-i} \]

\[ E(z) \rightarrow \begin{array}{c} D(z) \end{array} \rightarrow M(z) = D(z)\,E(z) \]
Discrete convolution

\[ x^*(t) = \sum_{k=0}^{\infty} x(kT) \delta(t - kT) \]

\[ y(t) = \begin{cases} 
  g(t)x(0) & 0 \leq t < T \\
  g(t)x(0) + g(t-T)x(T) & T \leq t < 2T \\
  g(t)x(0) + g(t-T)x(T) + g(t-2T)x(2T) & \ldots \\
  \vdots \\
  g(t)x(0) + g(t-T)x(T) + \ldots + g(t-kT)x(kT) & \ldots 
\end{cases} \]
Discrete time systems

Since \( g(t) = 0, t < 0 \), we have

\[
y(t) = g(t)x(0) + g(t - T)x(T) + \ldots + g(t - kT)x(kT)
\]

\[
= \sum_{h=0}^{k} g(t - hT)x(hT) \quad 0 \leq t < (k + 1)T
\]

Considering the samples of \( y(t) \) \( t = kT, \ k = 0, 1, 2, \ldots \), we have

\[
y(kT) = \sum_{h=0}^{k} g(kT - hT)x(hT)
\]

\[
= \sum_{h=0}^{k} x(kT - hT)g(hT)
\]
Example: $x(t) = e^{-t}$ \[ G(s) = \frac{1}{1 + s} \quad T = 1 \]
Example (cont’d)
Case a)

\[ Y_a(s) = G(s)X(s) = \frac{1}{s + 1} \frac{1}{s + 1} = \frac{1}{(s + 1)^2} \]

\[ y_a(kT) = kT e^{-kT} \]
Example (cont’d)
Case b)

\[
y_b(t) = \begin{cases} 
  g(t)x(0) \\
  g(t)x(0) + g(t - T)x(T) \\
  \vdots \\
  g(t)x(0) + g(t - T)x(T) + \ldots + g(t - kT)x(kT) 
\end{cases}
\]

In this case \( g(t) = e^{-t} \) (inverse transform of \( G(s) \)), hence:

\[
y_b(t) = \begin{cases} 
  e^{-t} \\
  e^{-t} + e^{-(t-T)}e^{-T} = 2e^{-t} \\
  \vdots \\
  e^{-t} + \ldots + e^{-(t-kT)}e^{-kT} = (k + 1)e^{-t} 
\end{cases}
\]

\[
y_b(kT) = (k + 1)e^{-kT}
\]
Example (cont’d)
Case b)

(b) -- Risposta al segnale $x^*$
**Discrete transfer function**

\[
y(kT) = \sum_{h=0}^{\infty} g(kT - hT)x(hT)
\]

\[
X(z) \xrightarrow{G(z)} Y(z)
\]

\[
X(z) = \mathcal{Z}[x(kT)] = 1 \quad \rightarrow \quad Y(z) = G(z)
\]

**Discrete harmonic response function**

\[
G(e^{j\omega T}), \quad 0 \leq \omega \leq \frac{\pi}{T}
\]

\[
G(e^{j(\omega + k\omega_s)T}) = G(e^{j\omega T}), \quad G(e^{j(-\omega)T}) = G^*(e^{j\omega T})
\]
Discrete transfer function

The response of an asymptotically stable system \( G(z) \) wrt a sinusoidal input \( \sin(\omega k T) \) is, in steady state, a sinusoidal function \( A \sin(\omega k T + \varphi) \) whose amplitude \( A \) and phase \( \varphi \) are respectively given by the module and the phase of the vector \( G(e^{j\omega T}) \):

\[
A = |G(e^{j\omega T})| \quad \varphi = \text{Arg}[G(e^{j\omega T})]
\]

\( \mathcal{Z} \)-transform of the sinusoidal signal:

\[
X(z) = \mathcal{Z}[\sin(\omega t)] = \frac{z \sin \omega T}{z^2 - (2 \cos \omega T)z + 1} = \frac{1}{2j} \left( \frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}} \right)
\]

\[
Y(z) = G(z) X(z) = Y_0(z) + \frac{|G(e^{j\omega T})|}{2j} \left( \frac{e^{j\varphi} z}{z - e^{j\omega T}} - \frac{e^{j\varphi} z}{z - e^{-j\omega T}} \right)
\]

is the sum of the asymptotically vanishing term \( Y_0(z) \), corresponding to stable poles of \( G(z) \), and a sinusoidal term with amplitude and phase equal to \( |G(e^{j\omega T})| \) and \( \varphi = \text{Arg}[G(e^{j\omega T})] \) respectively.
Composing discrete transfer functions

**Case a)**

\[ x(t) \xrightarrow{\text{Fictitious sampler}} x^*(t) \xrightarrow{G(s)} y(t) \]

\[ Y(s) = G(s) X^*(s) \]

\[ Y^*(s) = [G(s) X^*(s)]^* = G(s)^* X^*(s) \]

\[ Y(z) = G(z) X(z) \]

**Case b)**

\[ x(t) \xrightarrow{\text{Fictitious sampler}} x^*(t) \xrightarrow{G(s)} y(t) \]

\[ Y(s) = G(s) X(s) \]

\[ Y^*(s) = [G(s) X(s)]^* \]

\[ Y(z) = \mathcal{Z}[G(s) X(s)] = GX(z) \neq G(z) X(z) \]
Composing discrete transfer functions

Example

\[
\begin{align*}
&x(t) \overset{\delta_T}{\rightarrow} x^*(t) \quad \overset{\delta_T}{\rightarrow} \quad \frac{1}{s+a} \quad u(t) \quad \overset{\delta_T}{\rightarrow} \quad \frac{1}{s+b} \quad y(t) \\
&G(s) \quad H(s)
\end{align*}
\]

\(a\)

Case a)

\[
\frac{Y(z)}{X(z)} = G(z) H(z) = \mathcal{Z}\left[\frac{1}{s+a}\right] \mathcal{Z}\left[\frac{1}{s+b}\right]
\]

Case b)

\[
\frac{Y(z)}{X(z)} = \mathcal{Z}[G(s) H(s)] = \mathcal{Z}\left[\frac{1}{s+a} \quad \frac{1}{s+b}\right]
\]
Feedback interconnection:

\[ R(s) \rightarrow E(s) \rightarrow E^*(s) \rightarrow G(s) \rightarrow C(s) \]

\[ \rightarrow H(s) \]

\[ \begin{align*}
E(s) &= R(s) - H(s) C(s) \\
E(s) &= R(s) - H(s) G(s) E^*(s)
\end{align*} \]

Sampling:

\[ \begin{align*}
E^*(s) &= R^*(s) - GH^*(s) E^*(s) \\
C^*(s) &= G^*(s) E^*(s)
\end{align*} \]

\[ C^*(s) = \frac{G^*(s) R^*(s)}{1 + GH^*(s)} \quad \rightarrow \quad C(z) = \frac{G(z) R(z)}{1 + GH(z)} \]

The discrete transfer function of the sampled system is:

\[ \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \]
Composing discrete transfer functions

Typical feedback interconnections

\[
C(z) = \frac{G(z) R(z)}{1 + GH(z)}
\]
Composing discrete transfer functions

\[ R(s) \quad \rightarrow \quad G_1(s) \quad \rightarrow \quad H(s) \quad \rightarrow \quad G_2(s) \quad \rightarrow \quad C(s) \quad \rightarrow \quad C'(z) \]
\[ C(z) = \frac{G_1(z) G_2(z) R(z)}{1 + G_1(z) G_2 H(z)} \]

\[ R(s) \quad \rightarrow \quad G_1(s) \quad \rightarrow \quad H(s) \quad \rightarrow \quad G_2(s) \quad \rightarrow \quad C(s) \quad \rightarrow \quad C'(z) \]
\[ C(z) = \frac{G_2(z) G_1 R(z)}{1 + G_1 G_2 H(z)} \]
Composing discrete transfer functions

\[ C'(z) = \frac{G R(z)}{1 + G H(z)} \]
1. **Indirect method**: discretization of an analog controller
2. **Direct method**: using analytical methods in discrete time domain
3. **Standard controllers**: PID
DISCRETIZATION OF ANALOG CONTROLLERS
Three steps

1. Definition of sampling time $T$ and verification of the phase margin of the system

\[ H_0(s) = \frac{1 - e^{-sT}}{s} \approx \frac{T}{\frac{T}{2} s + 1} \]

\[ H_0(s) \approx e^{-sT/2} \]

\[
\begin{array}{c}
x(t) \quad e(t) \\
\downarrow \quad \downarrow \quad \downarrow \\
D(s) \quad 1 \quad G(s)
\end{array}
\]

2. Discretization of the analog controller $D(s)$

3. A posteriori simulative verification
Backward difference method

\[
D(z) = D(s) \bigg|_{s = \frac{1 - z^{-1}}{T}}
\]

Example:

\[
\frac{dy(t)}{dt} + ay(t) = ax(t)
\]

\[
\int_0^t \frac{dy(t)}{dt} dt = -a \int_0^t y(t) dt + a \int_0^t x(t) dt
\]

Evaluating for \(t = kT\), and for \(t = (k - 1)T\) and subtracting, we obtain

\[
y(kT) - y((k - 1)T) = -a \int_{(k-1)T}^{kT} y(t) dt + a \int_{(k-1)T}^{kT} x(t) dt \approx -aT [y(kT) - x(kT)]
\]

\[
Y(z) = z^{-1}Y(z) - aT [Y(z) - X(z)]
\]

\[
\frac{Y(z)}{X(z)} = G(z) = \frac{aT}{1 - z^{-1} + aT} = \frac{a}{1 - \frac{z^{-1}}{T}} + a
\]
Backward difference method

Integrazione all’indietro

$y(t)$

$t$
Correspondence between plane $s$ and plane $z$:

If $z = \sigma + j\omega$

$$Re \left( \frac{\sigma + j\omega - 1}{\sigma + j\omega} \right) = Re \left( \frac{(\sigma + j\omega - 1)(\sigma - j\omega)}{\sigma^2 + \omega^2} \right) = \frac{\sigma^2 - \sigma + \omega}{\sigma^2 + \omega^2} < 0$$

$$(\sigma - \frac{1}{2})^2 + \omega^2 < \left( \frac{1}{2} \right)^2$$

Stable controllers $D(s)$ are mapped in discrete time stable controllers $D(z)$. 

**Forward difference method**

\[ D(z) = D(s) \bigg|_{s = \frac{z - 1}{T}} \]

Example:

\[
\int_{(k-1)T}^{kT} y(t)dt \approx Ty((k-1)T), \quad \int_{(k-1)T}^{kT} x(t)dt \approx Tx((k-1)T)
\]

\[
y(kT) = y((k-1)T) - aT [y((k-1)T) - x((k-1)T)]
\]

\[
\frac{Y(z)}{X(z)} = G(z) = \frac{aT z^{-1}}{1 - (1 - aT)z^{-1}} = \frac{a}{1 - \frac{Tz^{-1} - 1}{Tz^{-1}} + a}
\]
**Forward difference method**

\[ Re(s) = Re \left( \frac{z - 1}{T} \right) < 0 \]

\[ \rightarrow \quad Re(z) < 1 \]

Stable controllers \( D(s) \) may be mapped in discrete time **unstable** controllers \( D(z) \)!!
**Bilinear transformation**

\[
D(z) = D(s)\bigg|_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}}
\]

also called **trapezoidal integration** (or Tustin’s transformation)

\[
\int_{(k-1)T}^{kT} y(t) \, dt \approx \frac{[y(kT) + y((k - 1)T)]T}{2}
\]

\[
\int_{(k-1)T}^{kT} x(t) \, dt \approx \frac{[x(kT) + x((k - 1)T)]T}{2}
\]
Bilinear transformation

\[ \Re \left( \frac{z - 1}{z + 1} \right) < 0 \]

\[ \Re \left( \frac{\sigma + j\omega - 1}{\sigma + j\omega + 1} \right) = \Re \left[ \frac{\sigma^2 - 1 + \omega^2 + j2\omega}{(\sigma + 1)^2 + \omega^2} \right] < 0 \]

\[ \sigma^2 + \omega^2 < 1 \]
Bilinear transformation

Frequential relation between \( w \) plane, \( z \) plane and \( s \) plane

\[
\begin{align*}
\text{Piano } w & : w = \frac{2}{T} \frac{z-1}{z+1} \\
\text{Piano } z & : z = e^{sT} \\
\text{Piano } s & : s = \frac{1}{T} \ln z
\end{align*}
\]

The transformation does not generate frequential overlapping but introduce distortions!!

\[
\begin{align*}
j \Omega & = \frac{2}{T} \frac{1 - e^{-j\omega T}}{1 + e^{-j\omega T}} = \frac{2}{T} \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{e^{j\omega T/2} + e^{-j\omega T/2}} \\
& = \frac{2}{T} \frac{2j \sin \omega T/2}{2 \cos \omega T/2} = j \frac{2}{T} \tan \frac{\omega T}{2}
\end{align*}
\]
Bilinear transformation

\[ j \Omega = j \frac{2}{T} \tan \frac{\omega T}{2} \]

\[ D_c(j \Omega) = D_d(e^{j \omega T}) \]

for

\[ \Omega = \frac{2}{T} \tan \frac{\omega T}{2} \]
Bilinear transformation with prewarping

\[
\begin{align*}
    s &= \frac{\omega_1}{\tan \frac{\omega_1 T}{2}} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{\omega_1}{\tan \frac{\omega_1 T}{2}} \frac{z - 1}{z + 1}
\end{align*}
\]

For \( \Omega = \omega_1 \) we have \( \omega = \omega_1 \)

Example

\[
G(s) = \frac{a}{s + a}
\]

Prewarping at \( \omega = a \)

\[
\begin{align*}
    s &= \frac{a}{\tan \frac{aT}{2}} \frac{1 - z^{-1}}{1 + z^{-1}} \\
    G_d(z) &= \frac{\tan \frac{aT}{2} (1 + z^{-1})}{(\tan \frac{aT}{2} - 1)z^{-1} + (\tan \frac{aT}{2} + 1)}
\end{align*}
\]
Example

Design a discrete time low pass filter that approximate the frequential behavior for \( \omega \in [0, 10] \text{rad/s} \) del filtro analogico

\[
G(s) = \frac{10}{s + 10} \quad \text{with} \quad T = 0.2 \text{ s}
\]

\[
G_d(z) = \frac{10}{2 \frac{1-z^{-1}}{1+z^{-1}} + 10} = \frac{1+z^{-1}}{2}
\]

\[
G_d(e^{j\omega T}) = \frac{10}{j \frac{2}{T} \tan \frac{\omega T}{2} + 10} = \frac{1}{j \tan 0.1\omega + 1}
\]
Example (cont’d)

Using prewarping at $\omega = 10 \text{ rad/s}$, we get

$$G_d(z) = \frac{10 \tan \frac{10\pi}{2}}{1-z^{-1} + 10z^{-1}} = \frac{0.609(1 + z^{-1})}{1 + 0.218z^{-1}}$$
Zeros/Poles matching

Write $D(s)$ enlightening poles and zeros factors.
Transform each pole and zero as

$$(s + a) \rightarrow (1 - e^{-aT}z^{-1})$$

$$(s + a \pm jb) \rightarrow (1 - 2e^{-aT}\cos bt z^{-1} + e^{-2aT}z^{-2})$$

Introduce as many zeros in $z = -1$ as the relative degree
Adjust the low frequencies gain ($z = 1$) or the high frequency gain ($z = -1$)

Example

$$D(s) = \frac{s + b}{s + a}$$

$$D(z) = k \frac{z - e^{-bT}}{z - e^{-aT}}$$

$$D(z = 1) = k \frac{1 - e^{-bT}}{1 - e^{-aT}} = D(s = 0) = \frac{b}{a} \quad \left( k = \frac{b}{a} \frac{1 - e^{-aT}}{1 - e^{-bT}} \right)$$
Zeros/Poles matching

Example: High pass filter

\[ D(s) = \frac{s}{s + a} \]
\[ D(z) = k \frac{z - 1}{z - e^{-aT}} \quad k = \frac{1 + e^{-aT}}{2} \]

Example:

\[ D(s) = \frac{1}{(s + a)^2 + b^2} = \frac{1}{(s + a + jb)(s + a - jb)} \]

⇒ relative degree equal to 2

\[ D(z) = k \frac{(z + 1)^2}{z^2 - 2ze^{-aT} \cos bT + e^{-2aT}} \]
\[ k = \frac{1 - 2e^{-aT} \cos bT + e^{-2aT}}{4(a^2 + b^2)} \]
Discretization design example

Plant:

\[ G(s) = \frac{1}{s(s + 2)} \]

Feedback specification: \( \delta = 0.5 \ (S = 16.3\%) \) e \( T_a \leq 2 \ s \ (\alpha 2\%) \)

\[ \frac{4}{\delta \omega_n} = 2 \quad \Rightarrow \quad \omega_n = 4 \ \text{rad/s} \]

Sampling time \( T \):
- damped oscillations with period \( 2\pi / (\omega_n \sqrt{1 - \delta^2}) = 1.814 \ s \)
- we want 8-10 samples per period
- \( T = 0.2 \ s \)

Effect of the zero order hold

\[ H_0(s) = \frac{1 - e^{-sT}}{s} \approx G_h(s) = \frac{1}{Ts/2 + 1} = \frac{10}{s + 10} \]
An analog controller $D(s)$ that meets the specifications for the modified plant $G_m$ is

$$G_m(s) = G_h(s)G(s) \quad \Rightarrow \quad D(s) = 20.25\frac{s + 2}{s + 6.667}$$

Being $G_a(s) = D(s)G_h(s)G(s)$, the feedback transfer function

$$G_0(s) = \frac{G_a(s)}{1 + G_a(s)} = \frac{202.5}{s^3 + 16.667s^2 + 66.67s + 202.5}$$

has poles in $s = -12.665 \pm j3.462$ ($\delta = 0.5$, $\delta\omega_n = 2 \Rightarrow T_a = 2 \ s$)

Discretizing the controller using the zeros/poles matching method:

$$z = e^{-6.667T} = 0.2644, \quad z = e^{-2T} = 0.6703$$

$$D(z) = k\frac{z - 0.6703}{z - 0.2644}$$

$D(z = 1) = D(s = 0) \quad \Rightarrow \quad k = 13.57$

$$D(z) = 13.57\frac{z - 0.6703}{z - 0.2644}$$
Step response of the feedback continuous time system (left) and feedback poles position (right)
Step response of the feedback system with discrete time controller (a) and corresponding control action (b)
How to choose sample time

Performances
- disturbance rejection
- set-point tracking
- control effort
- delays and stability
- robustness

Costs
- computational burden
- speed of computation
- precision
How to choose sample time

The effects of $T$ on performances are:
- effects of destabilization grow when $T$ grows;
- information loss grows when $T$ grows;
- discretization accuracy grows when $T$ decrease;

The best choice is the higher value of $T$ that guarantees good performances in terms of:

1) Loss of information: $\omega_s > 2\omega_b$

2) Smooth dynamics without delays: $6 < \frac{\omega_s}{\omega_b} < 20$

3) Disturbance rejection efficacy: $\omega_s > 2\omega_r$

4) Antialiasing filter efficacy: $\frac{\omega_s}{\omega_b} \geq 20$
How to choose sample time

Some practical rules:

a) \[ T \leq \frac{\tau_{dom}}{10} \]

b) \[ T \leq \frac{\theta}{4} \]

c) \[ T < \frac{T_a}{10}, \quad \omega_s > 10 \omega_n \]
ANALYTICAL DESIGN OF DIGITAL CONTROLLERS
Analytical design: poles and zeros assignment

Controller:

$$\text{Contr.}$$

$$v(k)$$

$$T$$

$$u(k)$$

$$H_0(s)$$

$$G_p(s)$$

$$y$$

$$G_p(z) = \frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)}$$

where $$B(z)$$ and $$A(z)$$ have no common factors and have degree equal to $$m$$ and $$n$$ respectively with $$n \geq m$$

Controller:

$$R(z) U(z) = T(z) V(z) - S(z) Y(z)$$
Control action combine a feedforward action

\[ H_{ff}(z) = \frac{T(z)}{R(z)} \]

and a feedback action

\[ H_{fb}(z) = \frac{S(z)}{R(z)} \]

Causality implies that

\[ \text{grado}(R) \geq \text{grado}(T), \quad \text{grado}(R) \geq \text{grado}(S) \]

In practice

\[ \text{grado}(R) = \text{grado}(T) = \text{grado}(S) \]

or

\[ \text{grado}(R) = 1 + \text{grado}(T) = 1 + \text{grado}(S) \]
Analytical design: poles and zeros assignment

Closed loop system: \[ \frac{Y(z)}{V(z)} = \frac{BT}{AR + BS} \]

Specs: \[ G_m(z) = \frac{B_m(z)}{A_m(z)} \]

Design equation: \[ \frac{BT}{AR + BS} = \frac{B_m}{A_m} \quad \text{where} \]
\[ \text{grado}(A_m) - \text{grado}(B_m) \geq \text{grado}(A) - \text{grado}(B) \]
Considering internal stable dynamics ("observer dynamics") we define

\[ G_m(z) = \frac{A_0(z) B_m(z)}{A_0(z) A_m(z)} \]

In order to have small errors for low frequency disturbances, the gain function

\[ \left. \frac{B(z) S(z)}{A(z) R(z)} \right|_{z=e^{j\omega T}} \]

must be high for \( \omega \to 0 \).

We can use integral actions: \( R(z) = (z - 1)^q R_1(z) \)

**Problem:** design \( R, S \) and \( T \)

The cancellation among zeros of \( B \) and zeros of \( A R + B S \), i.e., poles of the closed loop system, must be limited to stable zeros

\[ B = B^+ B^- \]
\[ B_m = B^- B'_m \]
The presence of non minimum phase zeros in \( G_p(z) \) depends on \( T \). In fact for \( T \to 0 \)

\[
G_p(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} G(s)\right] \approx \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{1}{s^3}\right]
\]

For \( l \geq 3 \) we always have unstable zeros. If \( l = 3 \) (and for \( T \to 0 \))

\[
G_p(z) \approx \frac{T^3}{3!} \frac{z^{-1}(1 + 4 z^{-1} + z^{-2})}{(1 - z^{-1})^3}
\]

with a zero in \( z = -3.73 \).

The stable factor \( B^+ \) can be canceled by choosing \( R = B^+ R' \)

Rewriting the design equation:

\[
\frac{B^+ B^- T}{B^+(A R' + B^- S)} = \frac{B^- B'_m}{A_m}
\]

which is

\[
\frac{T}{A R' + B^- S} = \frac{B'_m}{A_m}
\]
Considering also the observer dynamics $A_0$, the two design equations became

$$AR' + B^- S = A_0 A_m$$

$$T = A_0 B'_m$$

The characteristic equation of the feedback loop is

$$AR + BS = B^+ A_0 A_m$$

whose roots are
- stable zeros of the plant ($B^+$)
- specification poles ($A_m$)
- poles of the “observer dynamics” ($A_0$)
The Diophantine equation

\[ AX + BY = C \]

Necessary and sufficient condition for the existence of a solution \((X, Y)\) is that the maximum common divider of \(A\) and \(B\) is a factor of \(C\). This is satisfied if \(A\) and \(B\) do not have common factors.

If \((X_0, Y_0)\) then exist infinite solutions

\[
\begin{align*}
X &= X_0 + QB \\
Y &= Y_0 - QA
\end{align*}
\]

Example

\[ 3x + 4y = 7 \]

with \(x\) and \(y\) integers

Particular solution: \(x_0 = y_0 = 1\)

General solution \((n\) integer number) :

\[
\begin{align*}
x &= x_0 + 4n \\
y &= y_0 - 3n
\end{align*}
\]
The Diophantine equation

Exists a unique solution if

\[ \text{degree}(X) < \text{degree}(B) \]

or

\[ \text{degree}(Y) < \text{degree}(A) \]

The solution of the Diophantine equation can be obtained by solving the linear equations system

\[
A(z) = z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m
\]

\[
B(z) = b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n
\]

\[
C(z) = c_0 z^p + c_1 z^{p-1} + c_2 z^{p-2} + \cdots + c_p
\]

For \(\text{degree}(Y) = m - 1\) and \(\text{degree}(X) = p - m\)

\[
Y(z) = y_0 z^{m-1} + y_1 z^{m-2} + \cdots + y_{m-1}
\]

\[
X(z) = x_0 z^{p-m} + x_1 z^{p-m-1} + \cdots + x_{p-m}
\]

the system is squared and with a degree equal to \(p + 1\)
The Diophantine equation

\[
\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
{a_1} & 1 & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
{a_2} & {a_1} & \ddots & 0 & {b_1} & {b_0} & \ddots & 0 \\
\vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & 0 \\
{a_m} & \vdots & \vdots & \vdots & \vdots & {a_1} & {b_n} & \vdots & {b_0} \\
0 & {a_m} & \vdots & \vdots & \vdots & 0 & {b_n} & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & {a_m} & 0 & \ldots & 0 & {b_n}
\end{bmatrix}
\begin{bmatrix}
{x_0} \\
{x_1} \\
\vdots \\
x_{p-m} \\
y_0 \\
y_1 \\
\vdots \\
y_{m-1}
\end{bmatrix}
= 
\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{p-2} \\
c_{p-1} \\
\vdots \\
c_p
\end{bmatrix}
\]
Being \( x_0 = c_0 \), we have a reduced order system \( p \)

\[
\begin{bmatrix}
1 & 0 & \ldots & 0 & b_0 & 0 & \ldots & 0 \\
a_1 & 1 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\
a_2 & a_1 & \ddots & 0 & b_2 & b_1 & \ddots & 0 \\
\vdots & \vdots & \ddots & 1 & \vdots & \vdots & \ddots & b_0 \\
a_m & \vdots & a_1 & b_n & \vdots & b_1 \\
0 & a_m & \vdots & 0 & b_n & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & \ldots & 0 & a_m & 0 & \ldots & 0 & b_n
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_{p-m} \\
y_0 \\
y_1 \\
\vdots \\
y_{m-1}
\end{bmatrix}
= 
\begin{bmatrix}
c_1 - c_0 a_1 \\
c_2 - c_0 a_2 \\
\vdots \\
c_m - c_0 a_m \\
c_{m+1} \\
\vdots \\
c_p
\end{bmatrix}
\]

⇒ Sylvester’s matrix

In our application, in order to have a unique solution \( \text{degree}(S) = \text{degree}(A) - 1 \)
and for causality

\[
\text{degree}(A_m) - \text{degree}(B_m) \geq \text{degree}(A) - \text{degree}(B)
\]

\[
\text{degree}(A_0) \geq 2 \text{degree}(A) - \text{degree}(A_m) - \text{degree}(B^+) - 1
\]
1. Inputs $G_p = B/A$, $A_0$ and $G_m = B_m/A_m$

2. Decompose $B$

$$B = B^- B^+ \quad \quad B_m = B^- B'_m$$

where $B^+$ ismonic

3. Solve

$$\left(z - 1\right)^q A R'_1 + B^- S = A_0 A_m$$

with

$$\text{grado}(S) = \text{grado}(A) + q - 1$$

$$\text{grado}(R'_1) = \text{grado}(A_0) + \text{grado}(A_m) - \text{grado}(A) - q$$

4. write the control law

$$R u = T v - S y$$

con

$$R = B^+ R', \quad T = B'_m A_0, \quad R' = \left(z - 1\right)^q R'_1$$
Chose \( G_m = B_m/A_m \) as

\[
G_m(z) = \frac{Q(1) B^{-}(z)}{B^{-}(1) z^k Q(z)}
\]

where:

\[
Q(z) = z^2 + p_1 z + p_2
\]

\[
p_1 = -2e^{-\delta \omega_n T} \cos(\omega_n T \sqrt{1 - \delta^2})
\]

\[
p_2 = e^{-2\delta \omega_n T}
\]

or:

\[
Q(z) = z - a \\
a = e^{-T/\tau}
\]
Example: \[ G_p(s) = \frac{1}{s(s + 1)} \]

\[ G_p(z) = Z \left[ \frac{1 - e^{-sT}}{s} \frac{1}{s(s + 1)} \right] = \frac{K(z - b)}{(z - 1)(z - a)} \]

where

\[ a = e^{-T}, \quad K = a + T - 1, \quad b = 1 - \frac{T(1 - a)}{K} \]

Feedback system specification

\[ G_m(z) = \frac{z(1 + p_1 + p_2)}{z^2 + p_1 z + p_2} \]

\( G_p(z) \) has a zero in \( z = b \) which is not present in \( G_m(z) \) hence

\[ B = B^+ B^- \quad \text{with} \quad B^+ = z - b, \quad B^- = K \]
Example (cont’d) It must hold:

1. \( B'_m = \frac{B_m}{K} = \frac{z(1 + p_1 + p_2)}{K} \)
2. \( \text{degree}(A_0) \geq 0 \) and we choose \( A_0 = 1 \)
3. \( \text{degree}(R') = \text{degree}(A_0) + \text{degree}(A_m) - \text{degree}(A) = 0 \)
4. \( \text{degree}(S) = \text{degree}(A) - 1 = 1 \)

Hence \( R' = r_0 \) and \( S = (s_0z + s_1) \)

The design equation is

\[(z - 1)(z - a) r_0 + K (s_0z + s_1) = z^2 + p_1z + p_2\]

From which

\[r_0 = 1, \quad s_0 = \frac{1 + a + p_1}{K}, \quad s_1 = \frac{p_2 - a}{K}\]

Moreover

\[T(z) = A_0 B'_m = \frac{z(1 + p_1 + p_2)}{K} = t_0z\]
Example (cont’d) The control law $Ru = Tv - Sy$ is

$$u(k) = bu(k - 1) + t_0 v(k) - s_0 y(k) - s_1 y(k - 1)$$

Output $y(t)$ and control action $u(k)$ in case $\delta = 0.6$, $\omega_n = 1.2$ and $T = 0.2$
Output $y(t)$ and control action $u(k)$ in case $\delta = 0.6$, $\omega_n = 1.2$ and $T = 0.8$
Example (cont’d) To eliminate the “ringing” problem, let’s modify $G_m(z)$ as

$$G_m(z) = \frac{1 + p_1 + p_2}{1 - b} \frac{z - b}{z^2 + p_1 z + p_2}$$

from which: $B^+ = 1$, $B^- = K(z - b)$

$$B'_m = \frac{1 + p_1 + p_2}{K(1 - b)}$$

Since

$$\text{degree}(A_0) \geq 2\text{degree}(A) - \text{degree}(A_m) - \text{degree}(B^+) - 1 = 1$$

we chose $A_0(z) = z$. Moreover:

$$\text{degree}(R) = \text{degree}(A_m) + \text{degree}(A_0) - \text{degree}(A_m) = 1$$
$$\text{degree}(S) = \text{degree}(A) - 1 = 1$$
Example (cont’d) The design equation is

\[(z - 1)(z - a)(z + r_1) + K(z - b)(s_0z + s_1) = z^3 + p_1z^2 + p_2z\]

from which

\[r_1 = -b + \frac{b(b^2 + p_1b + p_2)}{(b - 1)(b - a)}\]

\[K(1 - b)(s_0 + s_1) = 1 + p_1 + p_2\]

\[K(a - b)(s_0a + s_1) = a^3 + p_1a^2 + p_2a\]

Solving

\[s_0 = \frac{\alpha_1 - \alpha_2}{1 - a}\]

\[s_1 = \frac{\alpha_2 - \alpha_1 a}{1 - a}\]

\[\alpha_1 = \frac{1 + p_1 + p_2}{K(1 - b)}\]

\[\alpha_2 = \frac{a^3 + p_1a^2 + p_2a}{K(a - b)}\]

Since: \(T(z) = A_0B'_m = z\frac{1 + p_1 + p_2}{k(1 - b)} = t_0z\), the resulting control law is:

\[u(k) = -r_1u(k - 1) + t_0v(k) - s_0y(k) - s_1y(k - 1)\]
Example (cont’d) If \( \delta = 0.6, \omega_n = 1.2 \) and \( T = 0.2 \) we obtain
Poles and zeros assignment: dealing with noises

\[
x = \frac{TB}{RA + BS} v + \frac{RB}{RA + BS} d_1 - \frac{SB}{RA + BS} d_2
\]

\[
x = \frac{TB}{RA + BS} v + \frac{B}{A} \frac{S}{R} B d_1 - \frac{S}{R} B d_2
\]
Poles and zeros assignment: dealing with noises

Defining:

\[ H_{fb} = \frac{S}{R} \text{ feedback gain} \]

\[ H_a = \frac{B}{A} \frac{S}{R} \text{ loop gain} \]

we obtain

\[ x = \frac{B_m}{A_m} v + \frac{H_a}{1 + H_a} \frac{1}{H_{fb}} d_1 - \frac{H_a}{1 + H_a} d_2 \]

By substituting

\[ x = \frac{B_m}{A_m} v + \frac{R B}{B + A_0 A_m} d_1 - \frac{S B}{B + A_0 A_m} d_2 \]

\[ = \frac{B_m}{A_m} v + \frac{R B^-}{A_0 A_m} d_1 - \frac{S B^-}{A_0 A_m} d_2 \]
Analytical design: deadbeat controller

The controller has just one degree of freedom (feedback control)

\[
\frac{U(z)}{E(\hat{z})} = \frac{S(z)}{R(\hat{z})} = D(z)
\]

The deadbeat specifications, in case of step references, are

a) the output must reach its final value in minimum time

b) steady state error must be zero

c) no oscillations between samples
Analytical design: deadbeat controller

To satisfy the deadbeat specs we impose

\[ G_m(z) = \frac{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}{z^N} \]

i.e.,

\[ G_m(z) = a_0 + a_1 z^{-1} + \cdots + a_N z^{-N} \]

with \( N \geq n \), \( n \) degree of the denominator of \( G_p(z) \)

From

\[ \frac{D(z)G_p(z)}{1 + D(z)G_p(z)} = G_m(z) \]

we obtain

\[ D(z) = \frac{G_m(z)}{G_p(z)[1 - G_m(z)]} \]
Analytical design: deadbeat controller

Causality conditions:
1. $D(z)$ with positive relative degree
2. If $G_p(z)$ has a factor $z^{-k}$, $G_m(z)$ must have a factor $z^{-h}$ with $h \geq k$

Stability conditions:
1. All unstable poles of $G_p(z)$ must be zeros of $1 - G_m(z)$
2. All unstable zeros of $G_p(z)$ must be zeros of $G_m(z)$

We refer to reference signals

$$V(z) = \frac{P(z)}{(1 - z^{-1})^{q+1}}$$

- if $P(z) = 1$, $q = 0$ we have the unitary step
- if $P(z) = T z^{-1}$, $q = 1$ we have the unitary ramp
- if $P(z) = \frac{1}{2} T^2 z^{-1} (1 + z^{-1})$, $q = 2$ we have the parable $v(t) = \frac{1}{2} t^2$
Since

\[ E(z) = V(z) - Y(z) = V(z) [1 - G_m(z)] \]

\[ = \frac{P(z) [1 - G_m(z)]}{(1 - z^{-1})^{q+1}} \]

the error goes to zero in finite time and remains null if

\[ 1 - G_m(z) = (1 - z^{-1})^{q+1} N(z) \]

which is

\[ E(z) = P(z) N(z) \]

Hence the controller is given by

\[ D(z) = \frac{G_m(z)}{G_p(z)(1 - z^{-1})^{q+1} N(z)} \]
If $G_p(s)$ is stable, in order to avoid oscillations between samples ("ripple"), we ask for $t \geq nT$

\[
y(t) = \text{const. for step}
\]
\[
\dot{y}(t) = \text{const. for ramp}
\]
\[
\ddot{y}(t) = \text{const. for parable}
\]

These must be translated in conditions on the control.

For example, in case of step input, the control $u(t)$ must be constant in steady state.
Deadbeat controller: example

Design $D(z)$ so that the closed loop system has a deadbeat step response.
We choose $T = 0.8\ s$

$$G_p(s) = \frac{1}{s(s + 1)}$$

$$G_p(z) = \mathcal{Z}\left[\frac{1 - e^{-sT}}{s} \cdot \frac{1}{s(s + 1)}\right] = \frac{K(z - b)}{(z - 1)(z - a)} = \frac{K(1 - b z^{-1}) z^{-1}}{(1 - z^{-1})(1 - a z^{-1})}$$

$$= \frac{0.2493 (1 + 0.7669 z^{-1}) z^{-1}}{(1 - z^{-1})(1 - 0.4493 z^{-1})}$$
Since $G_p(z)$ has a delay $z^{-1}$ and $n = 2$, we choose

$$G_m(z) = a_1 z^{-1} + a_2 z^{-2}$$

Since the input is a step:

$$1 - G_m(z) = (1 - z^{-1}) N(z)$$

this let also to avoid the cancellation of the critical pole of $G_p(z)$ in $z = 1$.

To avoid ripple we impose that $c(t) = \text{cost}$ for $t \geq 2T$, which is guaranteed by $u(t) = \text{cost}$ for $t \geq 2T$, i.e.

$$U(z) = b_0 + b_1 z^{-1} + b(z^{-2} + z^{-3} + \ldots)$$

where $b = 0$ since $G_p(s)$ has an integral action.
Deadbeat controller: example

Therefore

\[ U(z) = b_0 + b_1 z^{-1} \]

Moreover

\[
U(z) = \frac{Y(z)}{G_p(z)} = \frac{Y(z)V(z)}{V(z)G_p(z)} = G_m(z) \frac{V(z)}{G_p(z)}
\]

\[
= G_m(z) \frac{1}{(1 - z^{-1})} \frac{(1 - z^{-1})(1 - 0.4493z^{-1})}{0.2493(1 + 0.7669z^{-1})z^{-1}}
\]

\[
= G_m(z) \frac{0.2493(1 + 0.7669z^{-1})z^{-1}}{(1 - 0.4493z^{-1})}
\]

Making equal

\[
G_m(z) = (1 + 0.7669z^{-1})z^{-1} G_1
\]

\[
U(z) = 4.01(1 - 0.4493z^{-1})G_1
\]

with \( G_1 = \text{cost} \).
From

\[ 1 - a_1 z^{-1} - a_2 z^{-2} = (1 - z^{-1})N(z) \]

we have

\[ N(z) = 1 + (1 - a_1)z^{-1} \quad 1 - a_1 - a_2 = 0 \]

Making equal we obtain

\[ G_1 = a_1, \quad a_2 - 0.7669a_1 = 0 \]

\[ a_1 = 0.566, \quad a_2 = 0.434 \]

And finally

\[ G_m(z) = 0.566z^{-1} + 0.434z^{-2} \]

\[ N(z) = 1 + 0.434z^{-1} \]

\[ D(z) = \frac{G_m(z)}{G_p(z)(1 - z^{-1})N(z)} = \frac{2.27 - 1.02z^{-1}}{1 + 0.434z^{-1}} \]
Deadbeat controller: example
In case the plant has an unmodelled dynamics, i.e.,

\[ G_p(s) = \frac{10}{s(s + 1)(s + 10)} \]
Simplified deadbeat controller

- Just for step input
- $G_p(z)$ is stable and minimum phase
- Controller cancels all system dynamics
- Simplified specs: $G_m(z) = z^{-k}$ with $k$ greater or equal to the intrinsic delay of $G_p(z)$
- The controller $D(z)$ results:

$$D(z) = \frac{1}{G_p(z)} \frac{z^{-k}}{1 - z^{-k}}$$